Integral trees with diameters 5 and 6 *

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Abstract
In this paper, some new families of integral trees with diameters 5 and 6 are constructed. All these classes are infinite. They are different from those in the existing literature. We also prove that the problem of finding integral trees of diameters 5 and 6 is equivalent to the problem of solving some Diophantine equations. The discovery of these integral trees is a new contribution to the search for such trees.

Key Words: Integral tree, Diameter, Diophantine equation.

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1 Introduction

The notion of integral graphs was first introduced by F. Harary and A.J. Schwenk in 1974 (see [7]). A graph G is called integral if all the zeros of the characteristic polynomial \( P(G, x) \) are integers. The 23rd open problem of reference [4] is about trees with purely integral eigenvalues. All integral trees with diameters less than 4 are given in [1, 4]. Results on integral trees with diameters 4, 5, 6 and 8 can be found in [1, 2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 20, 21, 22, 23, 24].

Various families of integral balanced trees were studied in [1, 4, 5, 7, 8, 9, 11, 12, 14, 19, 20, 21, 22]. A tree \( T \) is called balanced if the vertices at the same distance from the center of \( T \) have the same degree. Balanced trees split into two families according to the parity of the diameter. We shall code a balanced tree of diameter 2\( k \) by the sequence \( (n_k, n_{k-1}, \ldots, n_1) \) or the tree \( T(n_k, n_{k-1}, \ldots, n_1) \), where \( n_j \) \((j = 1, 2, \ldots, k)\) denotes the number of successors of a vertex at distance \( k - j \) from the center. Let the tree \( K_{1,s} \bullet T(n_k, n_{k-1}, \ldots, n_1) \) of diameter 2\( k \) be obtained by identifying the center \( w \) of \( K_{1,s} \) and the center \( v \) of \( T(n_k, n_{k-1}, \ldots, n_1) \). Let the tree \( T[m, r] \) of diameter 3 be formed by joining the centers of \( K_{1,m} \) and \( K_{1,r} \) with a new edge, and let the tree \( T'[m, r] \) of diameter 5 (or \( T'[r, m] \) of diameter 6) be obtained by attaching \( t \) new endpoints to each vertex of the tree \( T[m, r] \) (or \( T[r, m] \)). Integral trees of diameters 5 and 6 were studied in [1, 2, 9, 11, 12, 13, 14, 15] and [1, 2, 8, 9, 11, 12, 14, 15, 18, 19, 20, 21, 22].

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Infinitely many integral trees $T^t[m, r]$ of diameter 5 were first constructed by R.Y. Liu in [14]. Later Z.F. Cao obtained general results on these classes by using the solutions of some Pell equations in [2], and then Y. Li obtained more general results on these classes by using the solutions of some more general quadratic Diophantine equations in [13]. Integral trees $T^t(r, m)$, $T(r, m, t)$, and $K_{1,s} \times T(r, m, t)$ of diameter 6 were investigated in [1, 2, 8, 9, 11, 12, 14, 15, 18, 19, 20, 21, 22]. In this paper, some new families of integral trees with diameters 5 and 6 are constructed. All these classes are infinite. They are different from those in the existing literature. We also prove that the problem of finding integral trees of diameters 5 and 6 is equivalent to the problem of solving some Diophantine equations. The discovery of these integral trees is a new contribution to the search of such trees. We believe it is useful for constructing other integral trees.

Firstly, we shall give some lemmas on graphs, the first three of which can be found in [5]. For notations and terminology, we refer to [5].

**Lemma 1.1.** Let $G_1 \cup G_2$ denote the union of two disjoint graphs $G_1$ and $G_2$. If $u \in V(G_1)$, $v \in V(G_2)$ and $G = G_1 \cup G_2 + uv$, then

$$P(G, x) = P(G_1, x)P(G_2, x) - P(G_1 - u, x)P(G_2 - v, x).$$

**Lemma 1.2.** Let $G$ be a graph. If $u \in V(G)$, $v \notin V(G)$ and $G^* = G + uv$, then

$$P(G^*, x) = xP(G, x) - P(G - u, x).$$

**Lemma 1.3.** Let $G$ be a graph with $n$ vertices, and $G^t$ is obtained by attaching $t$ new endpoints to each vertex of the graph $G$. Then we have $P(G^t, x) = x^n(P(G, x) - \frac{1}{x})$.

The following Lemmas 1.4, 1.5 and 1.6 can be found in [6], [11] and [18], respectively.

**Lemma 1.4.** If $G \cdot H$ is the graph obtained from $G$ and $H$ by identifying the vertices $v \in V(G)$ and $w \in V(H)$, then

$$P(G \cdot H, x) = P(G, x)P(H_w, x) + P(G_v, x)P(H, x) - xP(G_v, x)P(H_w, x),$$

where $G_v$ and $H_w$ are the subgraphs of $G$ and $H$ induced by $V(G) \setminus \{v\}$ and $V(H) \setminus \{w\}$, respectively.

**Lemma 1.5.** (1) $P(K_{1,t}, x) = x^{t-1}(x^2 - t)$.

(2) $P(T(m, t), x) = x^{m(t-1)+1}(x^2 - t)^{m-1}[x^2 - (m + t)]$.

**Lemma 1.6.**

$$P[K_{1,s} \times T(m, t), x] = x^{m(t-1)+(s-1)}(x^2 - t)^{m-1}[x^4 - (m + t + s)x^2 + st].$$

A graph $G$ is called a rooted graph if one vertex $u$ of $G$ is distinguished from the rest. The distinguished vertex $u$ is called the root-vertex, or simply the root. Let $r * G$ be the graph formed by joining the roots of $r$ copies of $G$ to a new vertex $v$, and let $K_{1,r} \cdot G$ be the graph obtained by identifying the center $z$ of $K_{1,r}$ and the root $u$ of $G$. The following Lemmas 1.7 and 1.8 can be found in [20].
Lemma 1.7. $P(r \ast G, x) = P^{r-1}(G, x)[xP(G, x) - rP(G - u, x)]$.

Proof. It is easy to check the validity by Lemmas 1.1 and 1.2.  

Lemma 1.8. $P(K_{1,r} \cdot G, x) = x^{r-1}[xP(G, x) - rP(G - u, x)]$.

Proof. It is easy to check the validity by Lemmas 1.4 and 1.5.  

Secondly, we shall give some facts in number theory. For notations and terminology, we refer to [3, 17, 25].

Let $d$ be a positive integer but not a perfect square, $m \neq 0$, and $m$ an integer. We shall study the Diophantine equation

$$x^2 - dy^2 = m. \quad (1)$$

If $x_1, y_1$ is a solution of Eqn.(1), for convenience, then $x_1 + y_1 \sqrt{d}$ is also called a solution of Eqn.(1). Let $s + t\sqrt{d}$ be any solution of the Pell equation

$$x^2 - dy^2 = 1. \quad (2)$$

Clearly, we know that

$$(x_1 + y_1 \sqrt{d})(s + t\sqrt{d}) = x_1s + y_1td + (y_1s + x_1t)\sqrt{d}$$

is also a solution of Eqn.(1). Then this solution and $x_1 + y_1 \sqrt{d}$ are called associate. If two solutions $x_1 + y_1 \sqrt{d}$ and $x_2 + y_2 \sqrt{d}$ of Eqn.(1) are associate, then we denote them by $x_1 + y_1 \sqrt{d} \sim x_2 + y_2 \sqrt{d}$. It is easy to verify that the associate relation $\sim$ is an equivalence relation. Hence, if Eqn.(1) has solutions, then all the solutions can be classified by the associate relation. Any two solutions in the same associate class are associate each other, any two solutions not in the same class are not associate.

The following Lemmas 1.9, 1.10 and 1.11 can be found in [3] or [25].

Lemma 1.9. A necessary and sufficient condition for two solutions $x_1 + y_1 \sqrt{d}$ and $x_2 + y_2 \sqrt{d}$ of Eqn.(1) to be in the same associate class $K$ is that

$$x_1x_2 - dy_1y_2 \equiv 0 (mod|m|), \quad y_1x_2 - x_1y_2 \equiv 0 (mod|m|).$$

Let $x_1 + y_1 \sqrt{d}$ be any solution of Eqn.(1), by Lemma 1.9, we have that $-(x_1 + y_1 \sqrt{d}) \sim x_1 + y_1 \sqrt{d}, -(x_1 - y_1 \sqrt{d}) \sim x_1 - y_1 \sqrt{d}$. Let $K$ and $\bar{K}$ be any two associate classes of solutions of Eqn.(1). If any solution $x + y\sqrt{d} \in K$, then $x - y\sqrt{d} \in \bar{K}$. The converse is also true. Hence, $K$ and $\bar{K}$ are called conjugate classes. If $K = \bar{K}$, then this class is called an ambiguous class. Let $u_0 + v_0 \sqrt{d}$ be the fundamental solution of the associate class $K$, where $v_0$ is positive and has the least value in the class $K$. If the class $K$ is ambiguous, we can assume that $u_0 \geq 0$.

Lemma 1.10. Let $K$ be any associate class of solutions of Eqn.(1), and $u_0 + v_0 \sqrt{d}$ be the fundamental solution of the associate class $K$. Let $x_0 + y_0 \sqrt{d}$ be the fundamental solution of Eqn. (2). Then we have that

$$0 \leq v_0 \leq \begin{cases} \frac{m^{\frac{1}{2}}\sqrt{m}}{\sqrt{2(m^2+1)}}, & \text{if } m > 0, \\ \frac{m^{\frac{1}{2}}\sqrt{m}}{\sqrt{2(m^2-1)}}, & \text{if } m < 0. \end{cases} \quad (3)$$
\[0 \leq |u_0| \leq \begin{cases} \sqrt{\frac{1}{2}(x_0 + 1)m}, & \text{if } m > 0, \\ \sqrt{\frac{1}{2}(x_0 - 1)(-m)}, & \text{if } m < 0. \end{cases}\] (4)

Lemma 1.11. (i) Let \(d\) be a positive integer but not a perfect square, \(m \neq 0\), and \(m\) be an integer. Then there are only finitely many associate classes for Eqn. (1), and the fundamental solutions of all these classes can be found by finite steps from (3) and (4).

(ii) Let \(K\) be an associate class of solutions of Eqn. (1), and \(u_0 + v_0\sqrt{d}\) be the fundamental solution of the associate class \(K\). Then all solutions of the class \(K\) are given by
\[x + y\sqrt{d} = \pm (u_0 + v_0\sqrt{d})(x_0 + y_0\sqrt{d})^n,
\] where \(n\) is an integer, and \(x_0 + y_0\sqrt{d}\) is the fundamental solution of Eqn. (2).

(iii) If \(u_0\) and \(v_0\) satisfy (3) and (4) but are not solutions of Eqn. (1), then there is no solution for Eqn. (1).

The following Lemmas 1.12 and 1.13 can be found in [3, 16] or [17].

Lemma 1.12. Let \(d \ (> 1)\) be a positive integer but not a perfect square. Then there exist solutions for Eqn. (2), and all the positive integral solutions \(x_k, y_k\) of Eqn. (2) are given by
\[x_k + y_k\sqrt{d} = \varepsilon^k,\] (5)
for \(k = 1, 2, 3, \ldots\), where \(\varepsilon = x_0 + y_0\sqrt{d}\) is the least positive solution of Eqn. (2). Suppose that \(\overline{\varepsilon} = x_0 - y_0\sqrt{d}\). Then we have that \(\varepsilon\overline{\varepsilon} = 1\) and
\[x_k = \frac{\varepsilon^k + \overline{\varepsilon}^k}{2}, \quad y_k = \frac{\varepsilon^k - \overline{\varepsilon}^k}{2\sqrt{d}},\] (6)
for \(k = 1, 2, \ldots\).

Lemma 1.13. Let \(u, v\) be the least positive solution of Eqn. (2), where \(d\ (> 1)\) is a positive integer but not a perfect square. Then the Pell equation
\[x^2 - dy^2 = -1\] (7)
has solutions if and only if there exist positive integral solutions \(s\) and \(t\) for the equations
\[s^2 + dt^2 = u, \quad 2st = v,
\]
and moreover \(s\) and \(t\) are the least positive solutions of Eqn. (7).

The following Lemmas 1.14 and 1.15 can be found in [16] and [25], respectively.

Lemma 1.14. Suppose that Eqn. (7) is solvable. Let \(\rho = x_0 + y_0\sqrt{d}\) be the least positive solution of Eqn. (7), where \(d\ (> 1)\) is a positive integer but not a perfect square. Then we have the following results.

(1) All the positive integral solutions \(x_k, y_k\) of Eqn. (7) are given by
\[x_k + y_k\sqrt{d} = \rho^k,\] (8)
for \(k = 1, 3, 5, \ldots\)
(2) All the positive integral solutions \( x_k, y_k \) of Eqn.(2) are given by Eqn.(8) for \( k = 2, 4, 6, \cdots \)

(3) Let \( \overline{p} = x_0 - y_0\sqrt{d} \), then \( \overline{p}\overline{p} = -1 \), and \( x_k, y_k \) can be defined by

\[
x_k = \frac{\rho^k + \overline{p}^k}{2}, \quad y_k = \frac{\rho^k - \overline{p}^k}{2\sqrt{d}}, \quad k = 1, 2, \cdots
\]

(9)

**Lemma 1.15.** (1) Let \( d > 1 \) be a positive integer with square-free divisor, if there exist \( d_1 > 1 \) and \( d_2 \) such that \( d = d_1d_2 \) and the Diophantine equation

\[
d_1x^2 - d_2y^2 = 1
\]

has positive integral solutions, then \( d_1, d_2 \) are uniquely determined by \( d \).

(2) Let \( \varepsilon_1 = x_1\sqrt{d_1} + y_1\sqrt{d_2} \) be the least positive integral solution of Eqn.(10). Then all positive integral solutions \( x_n, y_n \) of Eqn.(10) are given by

\[
x_n\sqrt{d_1} + y_n\sqrt{d_2} = \varepsilon_1^n, \quad 2 \nmid n.
\]

(11)

(3) Let \( \varepsilon = x_1\sqrt{d_1} - y_1\sqrt{d_2} \). Then \( \varepsilon_1\varepsilon_1 = 1 \) and

\[
x_n = \frac{\varepsilon_1^n + \varepsilon_1}{2\sqrt{d_1}}, \quad y_n = \frac{\varepsilon_1^n - \varepsilon_1}{2\sqrt{d_2}}, \quad 2 \nmid n.
\]

(12)

\section{Integral trees of diameter 5}

In this section, we shall construct infinitely many new integral trees of diameter 5. They are different from those in the existing literature.

**Theorem 2.1.** Let the tree \([K_{1,s} \bullet T(m,t)] \odot T(q,r)\) of diameter 5 be obtained by joining the center \( u \) of \( K_{1,s} \bullet T(m,t) \) and the center \( v \) of \( T(q,r) \) with a new edge. Then the tree \([K_{1,s} \bullet T(m,t)] \odot T(q,r)\) of diameter 5 is integral if and only if the equation

\[
(x^2 - t)^{m-1}(x^2 - r)^{-1}(x^2 - (m + t + s + q + r + 1)x^4 + [st + (q + r)(m + t + s)]x^2 + t}\]

\( + r + t(x^2 - t(sq + sr + r)) \) = 0

has only integral roots.

**Proof.** Note that the vertex \( u \) is the center of the tree \( K_{1,s} \bullet T(m,t) \), and the vertex \( v \) is the center of the tree \( T(q,r) \). Suppose that

\[
G_1 = K_{1,s} \bullet T(m,t), \quad G_2 = T(q,r).
\]

Then, by Lemma 1.1 we know that

\[
P([K_{1,s} \bullet T(m,t)] \odot T(q,r), x) = P(K_{1,s} \bullet T(m,t), x)P(T(q,r), x) - x^sP^m(K_{1,t}, x)P^q(K_{1,r}, x)
\]

By Lemmas 1.5 and 1.6, we have

\[
P([K_{1,s} \bullet T(m,t)] \odot T(q,r), x) = x^{m(t-1) + q(r-1) + s(x^2 - t)^{m-1}(x^2 - r)^{-1}(x^2 - (q + r)(m + t + s)x^2 + st)}
\]

\[-(x^2 - t)(x^2 - r)\]

\[= x^{m(t-1) + q(r-1) + s(x^2 - t)^{m-1}(x^2 - r)^{-1}(x^2 - (m + t + s + q + r + 1)x^4 + [st + (q + r)(m + t + s)]x^2 + t(sq + sr + r))} + r + t(x^2 - t(sq + sr + r))\]

\]
Thus, the theorem is proved.

The following Corollary 2.2 can be found in [1].

**Corollary 2.2.** If \( s = 0 \), then the tree \([K_{1,0} \bullet T(m,t)] \oplus T(q,r) = T(m,t) \oplus T(q,r)\) is not an integral tree with diameter 5.

Now we assume that \( s > 0 \) throughout the whole paper.

**Corollary 2.3.** If \( q + r = t \), then the tree \([K_{1,s} \bullet T(m,t)] \oplus T(q,r)\) of diameter 5 is integral if and only if there exist natural numbers \( a \) and \( b \) such that \( x^4 - (m + t + s + 1)x^2 + st + r\) can be factored as \((x^2 - a^2)(x^2 - b^2)\), \( t \) is a perfect square, and either \( q = 1 \) or \( q > 1 \) and \( r \) is a perfect square.

**Proof.** It is easy to check the validity by Theorem 2.1.

**Corollary 2.4.** For the tree \([K_{1,s} \bullet T(m,t)] \oplus T(q,r)\) of diameter 5, if \( q + r = t \), we have the following results.

1. When \( q = 1 \), let \( d > 1 \) such that there exist positive integral solutions for Eqn.\((7)\). Then, all positive integral solutions \(x_{2k-1}, y_{2k-1}\) of Eqn.\((7)\) are defined by Eqn.\((9)\). If \( s = d - 1 \), \( m = a^2 + b^2 - y_{2k-1} - d \), \( t = y_{2k-1}^2 - 1 \) and \( q = 1 \), if \( a, b, \) and \( b \) are positive integers, then the tree \([K_{1,s} \bullet T(m,t)] \oplus T(q,r)\) is integral with diameter 5, and there are infinitely many such integral trees.

2. When \( q > 1 \), if \( s = dc^2, t = f^2y_k^2, q = f^2(y_k^2 - c^2) > 0, r = e^2f^2, m = a^2 + b^2 - f^2y_k^2 - dc^2 - 1 > 0, \) and \( ab = efx_k \), where \( a, b, d(>1), e, f \) and \( k \) are positive integers, and \( d \) is not a perfect square, and all positive integral solutions \( x_k, y_k \) of Eqn.\((2)\) are given by Eqn.\((6)\), then the tree \([K_{1,s} \bullet T(m,t)] \oplus T(q,r)\) is integral with diameter 5, and there are infinitely many such integral trees.

3. When \( q > 1 \), if \( t \) and \( r \) are perfect squares, \( q = t - r > 0, s = \frac{a^2b^2 - r}{t} > 0, m = a^2 + b^2 - \frac{a^2b^2 - r}{t} - t - 1 > 0, \) where \( s, m, t, q, r, a \) and \( b \) are positive integers, then the tree \([K_{1,s} \bullet T(m,t)] \oplus T(q,r)\) is integral with diameter 5.

**Proof.** Since \( q + r = t \), by Theorem 2.1 we get that

\[
P([K_{1,s} \bullet T(m,t)] \oplus T(q,r)), x) = x^{m(t-1)+q(r-1)+s(x^2 - t)^m(x^2 - r)} - 1 | x^4 - (m + t + s + 1)x^2 + st + r.\]

By Corollary 2.3, we know that the tree \([K_{1,s} \bullet T(m,t)] \oplus T(q,r)\) of diameter 5 (where \( q + r = t \)) is integral if and only if there exist positive integral solutions for the following Diophantine equations \((13)\) satisfying one of the following two conditions:

1. \( q = 1, t \) is a perfect square, that is, \( t = t_1^2 \).
2. \( q > 1, t \) and \( r \) are perfect squares, that is, \( t = t_1^2 \) and \( r = r_1^2 \).

\[
\begin{align*}
a^2b^2 & = st + r, \\
a^2 + b^2 & = m + t + s + 1,
\end{align*}
\]
(1) By Eqn.(13), condition (i) and \( q + r = t \), we get that
\[
a^2b^2 - (s + 1)t = -1.
\]
Assume that \( ab = x, s + 1 = d, t = t_1^2 = y^2 \). Then Eqn. (14) can be changed into Eqn.(7). Hence, by Lemmas 1.13 and 1.14, Eqn.(13) and Eqn.(14), it is easy to check the validity of (1) of Corollary 2.4.

(2) By Eqn.(13), condition (ii) and \( q + r = t \), we get that
\[
a^2b^2 - st = r \Rightarrow a^2b^2 - st_1^2 = r_1^2 \Rightarrow \frac{a^2b^2}{r_1^2} - \frac{st_1^2}{r_1^2} = 1
\]
Assume that \( r = r_1^2 = e^2f^2, s = d^2, t = t_1^2 = f^2y^2 \) and \( ab = ef, x, y \), where \( a, b, d(> 1) \), \( e, f \) and \( k \) are positive integers, and \( d \) is not a perfect square. Then Eqn.(15) can be changed into Eqn.(2). Thus, by Eqn.(13), condition (ii) and \( q + r = t \), and Lemmas 1.12 and 1.14, it is easy to check the validity of (2) of Corollary 2.4.

(3) It is easy to check the validity of (3) of Corollary 2.4 by Theorem 2.1 or Corollary 2.3. The proof is now complete.

Note that we obtain the smallest integral tree \([K_{1,2} \cdot T(3,4)] \odot T(3,1)\) of diameter 5 in this class. Its characteristic polynomial is \( P([K_{1,2} \cdot T(3,4)] \odot T(3,1), x) = x^{11}(x^2 - 1)^3(x^2 - 4)^3(x^2 - 9) \) with order 25.

For (3) of Corollary 2.4, we simply list some examples of integral trees \([K_{1,s} \cdot T(m,t)] \odot T(q,r)\) with diameter 5.

**Example 2.5.** When \( q + r = t, q > 1 \), let \( s, m, t, q, r, a \) and \( b \) be those positive integers of (3) of Corollary 2.4, given in the following items, where \( a_1, b_1, k, k_1, k_2 \) and \( l \) are positive integers. Then the tree \([K_{1,s} \cdot T(m,t)] \odot T(q,r)\) is integral with diameter 5.

(1) \( s = k^2(l^2 + 2), m = (l^2 - k^2)(l^2 - k^2 + 2) > 0, t = k^2l^2, q = k^2(l^2 - k^2) > 1, r = k^4, a = k^2 \) and \( b = l^2 + 1 \),

(2) \( s = k^2(l^2 + 2), m = k^2l^4 - 1 > 0, t = k^2l^2, q = k^2(l^2 - k^2) > 1, r = k^4, a = k \) and \( b = k(l^2 + 1) \),

(3) \( s = k^2(l^2 - 2) > 0, m = (l^2 - k^2)(l^2 - k^2 - 2) > 0, t = k^2l^2, q = k^2(l^2 - k^2) > 1, r = k^4, a = k^2 \) and \( b = l^2 - 1 > 0 \),

(4) \( s = k^2(l^2 - 2) > 0, m = k^2(l^2 - 2)^2 - 1 > 0, t = k^2l^2, q = k^2(l^2 - k^2) > 1, r = k^4, a = k \) and \( b = k(l^2 - 1) > 0 \),

(5) \( s = l^2 + 2, m = l^4 - k^2l^2 + l^2 + k^2 - 2 > 0, t = k^2l^2, q = k^2(l^2 - 1) > 1, r = k^2, a = k \) and \( b = l^2 + 1 \),

(6) \( s = l^2 - 2 > 0, m = l^4 - k^2l^2 - 3l^2 + k^2 + 2 > 0, t = k^2l^2, q = k^2(l^2 - 1) > 1, r = k^2, a = k \) and \( b = l^2 - 1 > 0 \),

(7) \( s = a_1^2b_1^2(l^2 + 2), m = k_1^2a_1^4 + k_2^2b_1^4(l^2 + 1)^2 - a_1^2b_1^2(l^2 + 2) - k_2^2k_1^2a_1^2b_1^2 + 2 > 0, t = k_1^2k_2^2a_1^2b_1^2l^2, q = k_1^2k_2^2a_1^2b_1^2(l^2 - a_1^2b_1^2) > 1, r = k_1^2k_2^2a_1^2b_1^2, a = k_1a_1^2 \) and \( b = k_2b_1^2(l^2 + 1) \),

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Proof. It is easy to check the validity by Corollary 2.3 or (3) of Corollary 2.4.

Corollary 2.6. If \( q + r \neq t \), then the tree \([K_{1,s} \bullet T(m,t)] \ominus T(q,r)\) of diameter 5 is integral if and only if there exist natural numbers \( a, b \) and \( c \) such that
\[
x^5 - (m + t + s + q + r + 1)x^4 + [st + (q + r)(m + t + s) + r + t]x^2 - t(sq + sr + r)
\]
can be factored as \((x^2 - a^2)(x^2 - b^2)(x^2 - c^2)\), and both of these conditions hold:
(i) Either \( m = 1 \) or \( m > 1 \) and \( t \) is a perfect square.
(ii) Either \( q = 1 \) or \( q > 1 \) and \( r \) is a perfect square.

Proof. It is easy to check the validity by Theorem 2.1.

Corollary 2.7. When \( q + r \neq t \), \( m = 1 \), \( q > 1 \), let \( a, b, c, s, m, t, q \) and \( r \) be those positive integers of Corollary 2.6, given in the following Table 1 (where \( a, b, c, s, m, t, q \) and \( r \) are obtained by computer searching, and \( 1 \leq a \leq 16, b \leq c \leq b + 10, q + r \neq t, m = 1 \) and \( q > 1 \)). Then the tree \([K_{1,s} \bullet T(m,t)] \ominus T(q,r)\) is integral with diameter 5.

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Table 1: Integral tree \([K_{1,s} \bullet T(m,t)] \ominus T(q,r)\) with diameter 5, where \( q + r \neq t \), \( m = 1 \) and \( q > 1 \).

Proof. It is easy to check the validity by Theorem 2.1 or Corollary 2.6.

Corollary 2.8. When \( q + r \neq t \), \( m > 1 \), \( q > 1 \), \( t \) and \( r \) are perfect squares, let \( a, b, c, s, m, t, q \) and \( r \) be those positive integers of Corollary 2.6, given in the following Table 2 (where \( a, b, c, s, m, t, q \) and \( r \) are obtained by computer searching, and \( 1 \leq a \leq 7, b \leq c \leq 9, b \leq c \leq 20, q + r \neq t, m > 1 \) and \( q > 1 \)). Then the tree \([K_{1,s} \bullet T(m,t)] \ominus T(q,r)\) is integral with diameter 5.

Proof. It is easy to check the validity by Theorem 2.1 or Corollary 2.6.

Remark 2.9. From Theorem 2.1, we know that it is important to find positive integral solutions of the following Diophantine equations (16) satisfying one of the following four conditions:
(i) \( m = 1 \) and \( q = 1 \).
(ii) \( m > 1, q = 1, t \) is a perfect square.
(iii) \( m = 1, q > 1, r \) is a perfect square.
Table 2: Integral tree $[K_{1,s} \bullet T(m,t)] \oplus T(q,r)$ with diameter 5, where $q + r \neq t$, $m > 1$ and $q > 1$.

(iv) $m > 1$, $q > 1$, $t$ and $r$ are perfect squares.

$$\begin{align*}
a^2 + b^2 + c^2 &= m + t + s + q + r + 1 \\
a^2b^2 + b^2c^2 + c^2a^2 &= st + (q + r)(m + t + s) + t + r \\
a^2b^2c^2 &= t(sq + sr + r)
\end{align*}$$

By Theorem 2.1, Corollaries 2.3, 2.4, 2.6, 2.7 and 2.8, we know that there exist infinitely many such integral trees $[K_{1,s} \bullet T(m,t)] \oplus T(q,r)$ of diameter 5. However, (i) when $m = 1$ and $q = 1$, $1 \leq a \leq 15$, $a \leq b \leq a + 15$, $b \leq c \leq b + 20$, (ii) when $m > 1$, $q = 1$, $q + r \neq t$, and $t$ is a perfect square, $1 \leq a \leq 5$, $a \leq b \leq a + 8$, $b \leq c \leq b + 10$, we have not found positive integral solutions of Eqn.(16) by computer searching.

Hence, we raise the following Question 2.10.

**Question 2.10.** Are there integral trees $[K_{1,s} \bullet T(m,t)] \oplus T(q,r)$ of diameter 5 with $m = 1$, $q = 1$ or $m > 1$, $q = 1$, $q + r \neq t$, $t$ a perfect square ?

From Corollary 2.7, we raise the following Question 2.11.

**Question 2.11.** Can we prove that there are infinitely many integral trees $[K_{1,s} \bullet T(m,t)] \oplus T(q,r)$ of diameter 5 with $q + r \neq t$, $m = 1$ and $q > 1$ ?

**Remark 2.12.** For integral tree $[K_{1,s} \bullet T(m,t)] \oplus T(q,r)$ of diameter 5, by analyzing Table 2, we can see the following result. If $q + r \neq t$, $m > 1$ and $q > 1$, $t$ and $r$ are perfect squares, then the problem of finding such integral trees is equivalent to the problem of solving Eqn.(16). In particular, we can also see that

$$\begin{align*}
[x^2 - (q + r)][x^4 - (m + t + s)x^2 + st] - (x^2 - t)(x^2 - r) \\
= [x^2 - (q + r)][x^2 - r - x^2 - t](x^2 - r) \\
= (x^2 - r)[x^4 - (q + r + s^2 + 1)x^2 + t + \frac{st(q + r)}{r}] \\
= (x^2 - r)(x^2 - a^2)(x^2 - b^2).
\end{align*}$$
Thus, when \( q + r \neq t, m > 1 \) and \( q > 1 \), \( t \) and \( r \) are perfect squares, that is, \( t = t_1^2 \) and \( r = r_1^2 \), we know that the problem is equivalent to solving the following Diophantine equations (17).

\[
\begin{align*}
\alpha^2 + \beta^2 &= q + r + \frac{st}{r} + 1 \\
\alpha^2 \beta^2 &= t + \frac{s(t+q)}{r} \\
m + t + s &= r + \frac{s}{r}
\end{align*}
\]

Hence, we raise the following Question 2.13.

**Question 2.13.** When \( q + r \neq t, m > 1 \) and \( q > 1 \), \( t \) and \( r \) are perfect squares, can we prove that there are infinitely many positive integral solutions for Eqn. (16) or Eqn.(17)? Moreover, can we find all positive integral solutions of Eqn. (16) or Eqn.(17)?

**Theorem 2.14.** Let the tree \([K_{1,s} \cdot T(m, t)] \oplus [K_{1,p} \cdot T(q, r)]\) of diameter 5 be obtained by joining the center \(u\) of \( K_{1,s} \cdot T(m, t)\) and the center \( w\) of \( K_{1,p} \cdot T(q, r)\) with a new edge. Then the characteristic polynomial of the tree \([K_{1,s} \cdot T(m, t)] \oplus [K_{1,p} \cdot T(q, r)]\) of diameter 5 is

\[
\begin{align*}
P_2(\lambda) &= \lambda^{m-1}(\lambda^2 - r)^q - 1 \bigl\{ x^8 - (m + t + s + p + q + r + 1)x^6 + [(m + t + s)(p + q + r) \\
&+ st + pr + t + r]x^4 - [st(p + q + r) + pr(m + t + s) + rt]x^2 + prst \bigr\} = 0
\end{align*}
\]

has only integral roots.

For the tree \([K_{1,s} \cdot T(m, t)] \oplus [K_{1,p} \cdot T(q, r)]\) of diameter 5, we can see the following results.

(i) If \( p = 0 \), then \([K_{1,s} \cdot T(m, t)] \oplus [K_{1,p} \cdot T(q, r)] = [K_{1,s} \cdot T(m, t)] \oplus T(q, r).\)

(ii) If \( s = p = r = t \), then \([K_{1,s} \cdot T(m, t)] \oplus [K_{1,p} \cdot T(q, r)] = T^t[m, q].\) Integral trees \(T^t[m, q]\) of diameter 5 were investigated in [2, 13, 14, 15]. We simply list some examples from [2, 13, 14, 15]. The following Example 2.16 can be found in [2, 13].

**Example 2.16.** (1) Let \( d > 1 \) be a positive integer but not a perfect square, and \( x_k, y_k \) be defined by Eqn.(6). If \( m = d(\frac{y_n - y_l}{2})^2 \), \( r = d(\frac{y_n + y_l}{2})^2 \), and \( t = \frac{x^2_{n+1} - x^2_l}{4t} \), where \( n + l > 0 \), \( n \) and \( l \) are even, then all \( T^t[m, r] \) are integral trees with diameter 5.

(2) Let \( d > 1 \) such that there exists positive integral solutions for Eqn.(7), and let \( x_k, y_k \) be defined by Eqn.(9). If \( m = d(x_n y_n - x_l y_l)^2 \), \( r = d(x_n y_n + x_l y_l)^2 \), and \( t = \frac{x^2_{n+1} - x^2_l}{4t} \), where \( n + l > 0 \), then all \( T^t[m, r] \) are integral trees with diameter 5.

(3) Let \( d > 1 \) be a positive integer with square-free divisor, \( d = d_1d_2 \), \( d_1 > 1 \) such that Eqn.(10) has positive integral solutions, and let \( x_k, y_k \) be defined by Eqn.(12). If \( m = dx_1^2y_1^2 \), \( r = dx_1^2y_1^2 \), and \( t = d_1^2(\frac{x^2_{n+1} - x^2_l}{4t})^2 \), where \( k \neq l, 2 \parallel kl \), then all \( T^t[m, r] \) are integral trees with diameter 5.

(4) Let \( d > 1 \) be a positive integer with square-free divisor, and \( x_k, y_k \) be defined by Eqn.(6). Let \( m = dx_k^2y_1^2 \), \( r = dx_k^2y_1^2 \), and \( t = (\frac{x^2_{n+1} - x^2_l}{4t})^2 \), where \( k \neq l, \varepsilon = x_0 + y_0\sqrt{d} \) is the least
positive solution of Eqn.(2). If \( 2 \nmid x_0 \) or \( 2 | x_0 \), and \( k \equiv l (\text{mod } 2) \), then all \( T'[m,r] \) are integral trees with diameter 5.

For the tree \( [K_{1,s} \cdot T(m,t)] \oplus [K_{1,p} \cdot T(q,r)] \) of diameter 5, we have only found such integral trees for the case that \( s = t = p = r \). Hence, we raise the following Question 2.17.

**Question 2.17.** Are there integral trees \( [K_{1,s} \cdot T(m,t)] \oplus [K_{1,p} \cdot T(q,r)] \) of diameter 5 for \( s, t, p \) and \( r \) which are not all equal?

### 3 Integral trees of diameter 6

In this section, we shall construct infinitely many new integral trees of diameter 6. They are different from those in the existing literature.

**Theorem 3.1.** Let the tree \( r \ast (K_{1,s} \cdot T(m,t)) \) of diameter 6 be obtained by joining the centers of \( r \) copies of \( K_{1,s} \cdot T(m,t) \) to a new vertex \( w \), and let the tree \( K_{1,q} \cdot [r \ast (K_{1,s} \cdot T(m,t))] \) of diameter 6 be obtained by identifying the center \( z \) of \( K_{1,q} \) and the root \( w \) of \( r \ast (K_{1,s} \cdot T(m,t)) \), where \( r > 1 \). Then their characteristic polynomials are as follows.

1. \[ P[r \ast (K_{1,s} \cdot T(m,t))], x] = x^{rm(t-1)+r(s-1)+1}(x^2 - t)r^{m-1}[(x^4 - (m + t + s)x^2 + st)x^2 + t(r + s)]. \]
2. \[ P[K_{1,q} \cdot [r \ast (K_{1,s} \cdot T(m,t))], x] = x^{rm(t-1)+r(s-1)+q-1}(x^2 - t)r^{m-1}[(x^4 - (m + t + s)x^2 + st)x^2 + t(r + s) + q(m + t + s)x^2 - qst]. \]

**Proof.** It is easy to check the validity by Lemmas 1.5, 1.6, 1.7 and 1.8.

The following (2) of Corollary 3.2 can be found in [2, 14] or [22].

**Corollary 3.2.** For the tree \( K_{1,q} \cdot [r \ast (K_{1,s} \cdot T(m,t))] \) of diameter 6, we have the following results.

1. \( q = t \), then \( P[K_{1-q} \cdot [r \ast (K_{1,s} \cdot T(m,t))], x] = x^{(rm+1)(t-1)+r(s-1)(x^2 - t)r^{m-1}+1}[x^4 - (m + t + s)x^2 + st]^{r-1}[x^4 - (m + t + s + r)x^2 + st]. \]
2. \( q = s = t \), then \( P[T'(r,m), x] = P[K_{1,q} \cdot [r \ast (K_{1,s} \cdot T(m,t))], x] = x^{(rm+r+1)(t-1)}[x^2 - t]^{r(m-1)+1}[(x^4 - (m + 2t)x^2 + t^2)]^{r-1}[x^4 - (m + 2t + r)x^2 + t^2]. \]

**Proof.** It is easy to check the validity by Theorem 3.1.

**Theorem 3.3.** (1) The tree \( r \ast (K_{1,s} \cdot T(m,t)) \) of diameter 6 is integral if and only if the equation

\[ (x^2 - t)^{r(m-1)}[x^4 - (m + t + s)x^2 + st]^{r-1}[x^4 - (m + t + s + r)x^2 + t(r + s)] = 0 \]

has only integral roots.

(2) The tree \( K_{1,q} \cdot [r \ast (K_{1,s} \cdot T(m,t))] \) of diameter 6 is integral if and only if the equation

\[ (x^2 - t)^{r(m-1)}[x^4 - (m + t + s)x^2 + st]^{r-1}[x^4 - (m + t + s + r + q)x^4 + [t(r + s) + q(m + t + s)x^2 - qst] = 0 \]

has only integral roots.
Proof. It is easy to check the validity by Theorem 3.1.

Theorem 3.4. For any positive integer $n$, we have the following results.

(1) If the tree $r \ast [K_{1,s} \bullet T(m,t)]$ of diameter 6 is integral, and $m > 1$, then the tree $(rn^2) \ast [K_{1,sn^2} \bullet T(mn^2,t(n^2))]$ of diameter 6 is integral, too.

(2) If the tree $K_{1,q} \bullet [r \ast (K_{1,s} \bullet T(m,t))]$ of diameter 6 is integral, and $m > 1$, then the tree $K_{1,qn^2} \bullet [(rn^2) \ast (K_{1,sn^2} \bullet T(mn^2,t(n^2))]$ of diameter 6 is integral, too.

Proof. It is easy to check the validity by Theorem 3.1.

Remark 3.5. Unfortunately, we have not found integral trees $r \ast (K_{1,s} \bullet T(m,t))$ of diameter 6. We believe that such integral trees do not exist.

Remark 3.6. For the tree $K_{1,q} \bullet [r \ast (K_{1,s} \bullet T(m,t))]$ of diameter 6, when $q = s = t$, integral trees $T^t(r,m) = K_{1,t} \bullet [r \ast (K_{1,t} \bullet T(m,t))]$ of diameter 6 were studied in [2, 15, 22]. Here, our results on integral tree $T^t(r,m) = K_{1,t} \bullet [r \ast (K_{1,t} \bullet T(m,t))]$ of diameter 6 are different from those of [2, 15, 22].

Corollary 3.7. For any positive integer $n$, we have the following results.

(1) Let $d (> 1)$ be a positive integer but not a perfect square, and $x_k, y_k$ be defined by Eqn. (6). If $m = (dy_k+iy_{k-l})^2, r = x_{2k}x_{2l}$, and $t = (x_k + iy_{k-l})^2$, where $k > l > 0$, $k$ and $l$ are positive integers, then $T^t(r,m)$ (see [2]) and $T^{tn^2}(rn^2,mn^2)$ are integral trees with diameter 6.

(2) Let $d (> 1)$ be a positive integer but not a perfect square, let Eqn. (7) have positive integral solutions, and let $x_k, y_k$ be defined by Eqn. (9). If $m = \begin{cases} (dy_k+iy_{k-l})^2, & \text{if } k \equiv l \pmod{2}, \\ (x_{k+l}x_{k-l})^2, & \text{if } k \not\equiv l \pmod{2}, \end{cases}$ then $T^t(r,m)$ (see [2]) and $T^{tn^2}(rn^2,mn^2)$ are integral trees with diameter 6.

Proof. It is easy to check the validity by Theorems 3.1, 3.3 and 3.4.

Corollary 3.8. (1) Let $a, b, k$ and $n$ be positive integers satisfying $b < a < \frac{b^2}{k}$ and $k \mid a$. If $m = (a^2 - b^2)^2, r = (\frac{a^2}{k} - ka)^2 - (a^2 - b^2)^2, t = a^2b^2$, then $T^{tn^2}(rn^2,mn^2)$ is an integral tree with diameter 6.

(2) Let $a, b$ and $n$ be positive integers satisfying $b < a < b^2$. If $m = (a^2 - b^2)^2, r = (ab^2 - a)^2 - (a^2 - b^2)^2, t = a^2b^2$, then $T^t(r,m)$ (see [15]) and $T^{tn^2}(rn^2,mn^2)$ are integral trees with diameter 6.

(3) Let $a, b, c, d$ and $n$ be positive integers. If $m = (a^2 - b^2)^2, r = (c^2 - d^2)^2 - (a^2 - b^2)^2 > 0, t = a^2b^2 = c^2d^2$, then $T^t(r,m)$ (see [22]) and $T^{tn^2}(rn^2,mn^2)$ are integral trees with diameter 6.
Proof. It is easy to check the validity by Theorems 3.1, 3.3 and 3.4. ■

Corollary 3.9. For the tree $K_{1,q} \bullet [r \ast (K_{1,s} \bullet T(m,t))]$ of diameter 6, let $n$ be a positive integer, we have the following results.

(1) If $q = t$, then the tree $K_{1,t} \bullet [r \ast (K_{1,s} \bullet T(m,t))]$ of diameter 6 is integral if and only if there exist natural numbers $a, b, c$ and $d$ such that $x^4 - (m + t + s)x^2 + st$ can be factored as $(x^2 - a^2)(x^2 - b^2)$, and $x^4 - (m + t + s + r)x^2 + st$ can be factored as $(x^2 - c^2)(x^2 - d^2)$.

(2) If $q = t \neq s$, and the tree $K_{1,t} \bullet [r \ast (K_{1,s} \bullet T(m,t))]$ of diameter 6 is integral, then the trees $K_{1,t,2} \bullet [(rn^2) \ast (K_{1,s,n^2} \bullet T(mn^2, tn^2))]$ and $K_{1,s,n^2} \bullet [(rn^2) \ast (K_{1,t,n^2} \bullet T(mn^2, sn^2))]$ are integral with diameter 6.

Proof. It is easy to check the validity by Theorem 3.1 and Corollary 3.2. ■

Corollary 3.10. For the tree $K_{1,q} \bullet [r \ast (K_{1,s} \bullet T(m,t))]$ of diameter 6, let $n$ be a positive integer, $a, b, c$ and $d$ be those of Corollary 3.9, $q, r$, $s, m$ and $t$ be positive integers in the following Tables 3 and 4 (where $a, b, c, d, q, r, s, m$ and $t$ are obtained by computer searching, and $1 \leq a \leq 10, 1 \leq b \leq a + 20, 1 \leq c \leq 20, c \leq d \leq c + 20$). Then we have the following results.

(1) If $q = t = s, q, r, s, m$ and $t$ are positive integers in the following Table 3, then $T^{tn^2}(rn^2, mn^2) = K_{1,t,n^2} \bullet [(rn^2) \ast (K_{1,s,n^2} \bullet T(mn^2, tn^2))]$ is an integral tree with diameter 6.

(2) If $q = t \neq s$, and $q, r, s, m$ and $t$ are positive integers in the following Table 4, then $K_{1,t,n^2} \bullet [(rn^2) \ast (K_{1,s,n^2} \bullet T(mn^2, tn^2))]$ and $K_{1,s,n^2} \bullet [(rn^2) \ast (K_{1,t,n^2} \bullet T(mn^2, sn^2))]$ are integral trees with diameter 6.

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Table 3: Integral tree $T^{tn^2}(rn^2, mn^2) = K_{1,t,n^2} \bullet [(rn^2) \ast (K_{1,s,n^2} \bullet T(mn^2, tn^2))]$ of diameter 6, where $n$ is a positive integer.

Proof. It is easy to check the validity by Theorem 3.1 or Corollary 3.9. ■

Acknowledgements

The authors would like to express their thanks to the referees for many detailed comments and suggestions, which are very helpful for improving the presentation of the manuscript.
Table 4: Integral tree $K_{1,tn^2} \bullet \left([rn^2] \ast (K_{1,sn^2} \bullet T(mn^2, tn^2))\right)$ of diameter 6 and integral tree $K_{1,sn^2} \bullet \left([rn^2] \ast (K_{1,tn^2} \bullet T(mn^2, sn^2))\right)$ of diameter 6, where $n$ is a positive integer.

References


