SYMPLECTIC GRAPHS AND THEIR AUTOMORPHISMS

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Abstract. The general symplectic graph $Sp(2\nu, q)$ is introduced. It is shown that $Sp(2\nu, q)$ is strongly regular. Its parameters are computed, its chromatic number and group of graph automorphisms are also determined.

1. Introduction

Let $F_q$ be a finite field of any characteristic and $\nu \geq 1$ an integer. Let $F_q^{(2\nu)} = \{(a_1, \ldots, a_{2\nu}) : a_i \in F_q, i = 1, \ldots, 2\nu\}$. be the $2\nu$-dimensional row vector space over $F_q$. For any $\alpha_1, \ldots, \alpha_n \in F_q^{(2\nu)}$, we denote the subspace of $F_q^{(2\nu)}$ generated by $\alpha_1, \ldots, \alpha_n$ by $[\alpha_1, \ldots, \alpha_n]$. Thus, if $\alpha \neq 0 \in F_q^{(2\nu)}$ then $[\alpha]$ is an one dimensional subspace of $F_q^{(2\nu)}$ and $[\alpha] = [k\alpha]$ for any $k \in F_q^* = F_q \setminus \{0\}$.

Let $K$ be a $2\nu \times 2\nu$ nonsingular alternate matrix over $F_q$. The symplectic graph relative to $K$ over $F_q$ is the graph with the set of one dimensional subspaces of $F_q^{(2\nu)}$ as its vertex set and with the adjacency defined by

$[\alpha] \sim [\beta]$ if and only if $\alpha K^t \beta \neq 0$, for any $\alpha \neq 0, \beta \neq 0 \in F_q^{(2\nu)}$, where $[\alpha] \sim [\beta]$ means that $[\alpha]$ and $[\beta]$ are adjacent. Since any two $2\nu \times 2\nu$ nonsingular alternate matrices over $F_q$ are cogredient, any two symplectic graphs relative to two different $2\nu \times 2\nu$ nonsingular alternate matrices over $F_q$ are isomorphic. Thus we can assume that

$K = \begin{pmatrix}
0 & 1 & \cdots & 0 & 1 \\
-1 & 0 & \cdots & -1 & 0 \\
0 & 1 & \cdots & 0 & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 1 & \cdots & 0 & 1
\end{pmatrix}_{2\nu \times 2\nu}$

and consider only the symplectic graph relative to the above $K$ over $F_q$, which will be denoted by $Sp(2\nu, q)$.

When $q = 2$, the special case $Sp(2\nu, 2)$ of the graph $Sp(2\nu, q)$ was studied previously by Rotman [4], Rotman and Weichsel [5], Godsil and Royle [2, 3], etc. In the present paper we study the general case $Sp(2\nu, q)$. In Section 2, we show that

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$Sp(2\nu, q)$ is strongly regular and compute its parameters. We also prove that the chromatic number of $Sp(2\nu, q)$ is $q^\nu + 1$. Section 3 is devoted to discuss the group of automorphisms $\text{Aut}(Sp(2\nu, q))$ of the graph. The structure of this group depends on $q$ and $\nu$. When $q = 2$, $\text{Aut}(Sp(2\nu, 2))$ is isomorphic to the symplectic group of degree $2\nu$ over $\mathbb{F}_2$. When $q > 2$, $\text{Aut}(Sp(2\nu, q))$ is the product of two subgroups which are identified clearly (cf. Theorem 3.4).

2. STRONGLY REGULARITY AND CHROMATIC NUMBERS OF SYMPLECTIC GRAPHS

For any subspace $V$ of $\mathbb{F}_q^{(2\nu)}$, we denote the subspace of $\mathbb{F}_q^{(2\nu)}$ formed by all $\beta \in \mathbb{F}_q^{(2\nu)}$ such that $\alpha K^\beta = 0$ for all $\alpha \in V$ by $V^\perp$. Then $[\alpha] \sim [\beta]$ if and only if $\beta \not\in [\alpha]^{-1}$.

Denote the vertex set of the graph $Sp(2\nu, q)$ by $V(\mathbb{F}_q^{(2\nu)})$. We first show that $Sp(2\nu, q)$ is strongly regular.

**Theorem 2.1.** $Sp(2\nu, q)$ is a strongly regular graph with parameters

$$\left(\frac{q^{2\nu} - 1}{q - 1}, q^{2\nu-1}, q^{2\nu-2}(q - 1), q^{2\nu-2}(q - 1)\right)$$

and eigenvalues $q^{2\nu-1}$, $q^{\nu-1}$ and $-q^{\nu-1}$.

**Proof.** As $|\mathbb{F}_q^{(2\nu)}| = q^{2\nu}$, it follows that $|V(\mathbb{F}_q^{(2\nu)})| = \frac{q^{2\nu} - 1}{q - 1}$. For any $[\alpha] \in V(\mathbb{F}_q^{(2\nu)})$, since $\text{dim}([\alpha]^{-1}) = 2\nu - 1$, we see that the degree of $[\alpha]$ which is just the number of one dimensional subspaces $[\beta]$ such that $\beta \not\in [\alpha]^{-1}$, is $\frac{q^{2\nu} - q^{2\nu-1}}{q - 1} = q^{2\nu-1}$.

Let $[\alpha], [\beta]$ be any two different vertices of $Sp(2\nu, q)$ which are adjacent with each other or not. Then $\text{dim}([\alpha], [\beta]^{-1}) = 2\nu - 2$. Note that a vertex $[\gamma]$ is adjacent with both $[\alpha]$ and $[\beta]$ is equivalent to that $[\gamma] \not\in [\alpha]^{-1} \cup [\beta]^{-1}$. But $||[\alpha]^{-1} \cup [\beta]^{-1}| = ||[\alpha]^{-1}| + ||[\beta]^{-1}| - ||[\alpha], [\beta]^{-1}|$.

Hence the number of vertices which are adjacent with both $[\alpha]$ and $[\beta]$ is $q^{2\nu - 2q^{2\nu-1} + q^{2\nu-2}} = q^{2\nu-2}(q - 1)$. Therefore $Sp(2\nu, q)$ is a strongly regular graph with parameter

$$\left(\frac{q^{2\nu} - 1}{q - 1}, q^{2\nu-1}, q^{2\nu-2}(q - 1), q^{2\nu-2}(q - 1)\right).$$

By the same arguments as in [3, Section 10.2], we get that the eigenvalues of $Sp(2\nu, q)$ are $q^{2\nu-1}$, $q^{\nu-1}$ and $-q^{\nu-1}$.

Let $n \geq 2$. We say that a graph $X$ is $n$-partite if there are subsets $X_1, \ldots, X_n$ of the vertex set $V(X)$ of $X$ such that $V(X) = X_1 \cup \cdots \cup X_n$, where $X_i \cap X_j = \emptyset$ for all $i \neq j$, and that there is no edge of $X$ joining two vertices of the same subset. We are going to show that $Sp(2\nu, q)$ is $(q^\nu + 1)$-partite. We need some results about subspaces of $\mathbb{F}_q^{(2\nu)}$. A subspace $V$ of $\mathbb{F}_q^{(2\nu)}$ is called totally isotropic if $V \subseteq V^\perp$. Then totally isotropic subspaces of $\mathbb{F}_q^{(2\nu)}$ are of dimension $\leq \nu$ and there exist totally isotropic subspaces of dimension $\nu$ which are called maximal totally isotropic subspaces, cf. [6, Corollary 3.8].

The following lemma is due to Dye[1].
Lemma 2.2. There exist maximal totally isotropic subspaces $V_i$, $i = 1, \ldots, q'' + 1$, of $F_q^{(2\nu)}$ such that
\[
F_q^{(2\nu)} = V_1 \cup \cdots \cup V_{q''+1},
\]
where $V_i \cap V_j = \{0\}$ for all $i \neq j$.

Proposition 2.3. $Sp(2\nu, q)$ is $(q'' + 1)$-partite. That is, there exist subsets $X_1, \ldots, X_{q''+1}$ of $V(Sp(2\nu, q))$ such that
\[
V(Sp(2\nu, q)) = X_1 \cup \cdots \cup X_{q''+1},
\]
where $X_i \cap X_j = \emptyset$ for all $i \neq j$, and there is no edge of $Sp(2\nu, q)$ joining two vertices of the same subset. Moreover, the subsets $X_1, \ldots, X_{q''+1}$ can be so chosen that for any two distinct indices $i$ and $j$, every $\alpha \in X_i$ is adjacent with exactly $q'' - 1$ vertices in $X_j$.

Proof. Let $F_q^{(2\nu)} = V_1 \cup \cdots \cup V_{q''+1}$ as in 2.2. Set $X_i = \{[\alpha] : \alpha \neq 0 \in V_i\}$, $i = 1, \ldots, q'' + 1$. Then
\[
V(Sp(2\nu, q)) = X_1 \cup \cdots \cup X_{q''+1}, \quad X_i \cap X_j = \emptyset, \quad \text{for all } i \neq j.
\]
As $V_i$ is totally isotropic, we see that there is no edge joining any two vertices in $X_i$. Thus $Sp(2\nu, q)$ is $(q'' + 1)$-partite. For any $i \neq j$, let $[\alpha] \in X_i$. Since $V_j$ is maximal totally isotropic of dimension $\nu$, it follows that $\alpha \notin V_j = V_j^\perp$ and $\dim([\alpha]^\perp \cap V_j) = \dim([\alpha, V_j^\perp]) = \nu - 1$. Note that, for any $[\beta] \in X_j$, $[\beta]$ is adjacent with $[\alpha]$ if and only if $\beta \in V_j \setminus ([\alpha]^\perp \cap V_j)$. Hence the number of vertices in $X_j$ which is adjacent with $[\alpha]$ is $\frac{q'' - 1}{q-1} - \frac{q'' - 1 - 1}{q-1} = q'' - 1$.

Now we can compute the chromatic number of $Sp(2\nu, q)$.

Theorem 2.4. $\chi(Sp(2\nu, q)) = q'' + 1$.

Proof. By 2.3, we see that $\chi(Sp(2\nu, q)) \leq q'' + 1$. Note that $\chi(Sp(2\nu, q))$ is the minimal $n$ such that $Sp(2\nu, q)$ is $n$-partite. Suppose that $Sp(2\nu, q)$ is $n$-partite. Then there exist subsets $Y_1, \ldots, Y_n$ of $V(Sp(2\nu, q))$ such that
\[
V(Sp(2\nu, q)) = Y_1 \cup \cdots \cup Y_n, \quad Y_i \cap Y_j = \emptyset, \quad \text{for all } i \neq j,
\]
and there is no edge joining any two vertices in the same $Y_i$ for $i = 1, \ldots, n$. We want to show that $n \geq q'' + 1$. Suppose that $n < q'' + 1$. From the above equality, we have
\[
\sum_{i=1}^n |Y_i| = \frac{q'' - 1}{q-1} = \left(\frac{q - 1}{q-1}\right)(q'' + 1).
\]
Then there exists some $i$ such that $|Y_i| > \frac{q'' - 1}{q-1}$. Let $W_i$ be the subspace of $F_q^{(2\nu)}$ generated by all $\alpha$ such that $[\alpha] \in Y_i$. Then $W_i$ is a totally isotropic subspace, hence $dim W_i \leq \nu$. This turns out $|Y_i| \leq \frac{q'' - 1}{q-1}$, a contradiction. Hence $\chi(Sp(2\nu, q)) = q'' + 1$. \qed
3. Automorphisms of Symplectic Graphs

We recall that a $2\nu \times 2\nu$ matrix $T$ is called a symplectic matrix (or generalized symplectic matrix) of order $2\nu$ over $\mathbb{F}_q$ if $TK'T = K$ (or $TK'T = kK$ for some $k \in \mathbb{F}_q^*$, respectively). The set of symplectic matrices (or generalized symplectic matrices) of order $2\nu$ over $\mathbb{F}_q$ forms a group with respect to the matrix multiplication, which is called the symplectic group (or generalized symplectic group, respectively) of degree $2\nu$ over $\mathbb{F}_q$ and denoted by $Sp_{2\nu}(\mathbb{F}_q)$ (or $GSp_{2\nu}(\mathbb{F}_q)$). The center of $Sp_{2\nu}(\mathbb{F}_q)$ consists of the identity matrix $E$ and $-E$, and the factor group $Sp_{2\nu}(\mathbb{F}_q)/\{E, -E\}$ is called the projective symplectic group of degree $2\nu$ over $\mathbb{F}_q$ and denoted by $PSp_{2\nu}(\mathbb{F}_q)$. The center of $GSp_{2\nu}(\mathbb{F}_q)$ consists of all $kE$, where $k \in \mathbb{F}_q^*$, and the factor group of $GSp_{2\nu}(\mathbb{F}_q)$ with respect to its center is called the projective generalized symplectic group of degree $2\nu$ over $\mathbb{F}_q$ and denoted by $PGSp_{2\nu}(\mathbb{F}_q)$. Clearly, $PGSp_{2\nu}(\mathbb{F}_q) \cong PSp_{2\nu}(\mathbb{F}_q)$, and when $q = 2$, $GSp_{2\nu}(\mathbb{F}_2) = Sp_{2\nu}(\mathbb{F}_2)$.

**Proposition 3.1.** Let $T$ be a $2\nu \times 2\nu$ nonsingular matrix over $\mathbb{F}_q$ and

$$\sigma_T: V(Sp(2\nu, q)) \rightarrow V(Sp(2\nu, q))$$

$$[\alpha] \mapsto [\alpha T].$$

Then

1. $T \in GSp_{2\nu}(\mathbb{F}_q)$ if and only if $\sigma_T \in \text{Aut}(Sp(2\nu, q))$. In particular, when $q = 2$, $T \in Sp_{2\nu}(\mathbb{F}_2)$ if and only if $\sigma_T \in \text{Aut}(Sp(2\nu, 2))$

2. For any $T_1, T_2 \in GSp_{2\nu}(\mathbb{F}_q)$, $\sigma_{T_1} = \sigma_{T_2}$ if and only if $T_1 = kT_2$ for some $k \in \mathbb{F}_q*$.

**Proof.** It is clear that $\sigma_T$ is an one-one correspondence from $V(Sp(2\nu, q))$ to itself.

1. First assume $T \in GSp_{2\nu}(\mathbb{F}_q)$. Then $TK'T = kK$ for some $k \in \mathbb{F}_q^*$. For any $[\alpha], [\beta] \in V(Sp(2\nu, q))$, since $\alpha K'\beta = k^{-1}(\alpha T)K'(\beta T)$, $[\alpha] \sim [\beta]$ if and only if $\sigma_T([\alpha]) \sim \sigma_T([\beta])$, hence $\sigma_T \in \text{Aut}(Sp(2\nu, q))$.

Conversely, assume $\sigma_T \in \text{Aut}(Sp(2\nu, q))$. Then, for any $\alpha, \beta \neq 0 \in \mathbb{F}_q^{2\nu}$, $\alpha K'\beta = 0$ if and only if $\alpha (TK'T)\beta = 0$. Hence, for any $\alpha \neq 0 \in \mathbb{F}_q^{2\nu}$, the two systems of linear equations $(\alpha K)'X = 0$, $(\alpha TK'T)'X = 0$ have the same solutions. But $\text{rank}(\alpha K') = \text{rank}(\alpha TK'T) = 1$, we see that $\alpha K = k(\alpha TK'T)$ for some $k \in \mathbb{F}_q^*$, which depends on $\alpha$. Take $\alpha = (1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$, we get that $K = \text{diag}(k_1, k_2, \ldots, k_{2\nu})TK'T$, for some $k_1, k_2, \ldots, k_{2\nu} \in \mathbb{F}_q^*$. Take $\alpha = (1, 1, \ldots, 1)$, we see that $k_1 = k_2 = \ldots = k_{2\nu}$, hence $K = k_1TK'T$.

2. It is clear that $\sigma_{T_1} = \sigma_{T_2}$ if $T_1 = kT_2$ for some $k \in \mathbb{F}_q^*$. Conversely, suppose that $\sigma_{T_1} = \sigma_{T_2}$. Then, for any $\alpha \neq 0 \in \mathbb{F}_q^{2\nu}$, $\alpha T_1 = k\alpha T_2$ for some $k \in \mathbb{F}_q^*$. Take $\alpha = (1, 0, \ldots, 0), (0, 1, \ldots, 0)$, and so on as above, we see that $T_1 = kT_2$ for some $k \in \mathbb{F}_q^*$. □

By 3.1, every generalized symplectic matrix in $GSp_{2\nu}(\mathbb{F}_q)$ induces an automorphism of $Sp(2\nu, q)$ and two generalized symplectic matrices $T_1$ and $T_2$ induce the same automorphism of $Sp(2\nu, q)$ if and only if $T_1 = kT_2$ for some $k \in \mathbb{F}_q$. Thus $PSp_{2\nu}(\mathbb{F}_q)$ can be regarded as a subgroup of $\text{Aut}(Sp(2\nu, q))$. 

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Proposition 3.2. $Sp(2\nu, q)$ is vertex transitive and edge transitive.

Proof. For any $[\alpha], [\beta] \in V(Sp(2\nu, q))$, there exists $T \in Sp_{2\nu}(\mathbb{F}_q)$ such that $\alpha T = \beta$ by [6, Lemma 3.11]. Then $\sigma_T \in \text{Aut}(Sp(2\nu, q))$ such that $\sigma_T([\alpha]) = [\beta]$. Hence $Sp(2\nu, q)$ is vertex transitive.

Let $[\alpha_1], [\alpha_2], [\beta_1], [\beta_2] \in V(Sp(2\nu, q))$ such that $[\alpha_1] \sim [\alpha_2]$ and $[\beta_1] \sim [\beta_2]$. We may assume that $\alpha_1 K' \alpha_2 = \beta_1 K' \beta_2$. Then, by [6, Lemma 3.11] again, there exists $T \in Sp_{2\nu}(\mathbb{F}_q)$ such that $\alpha_1 T = \beta_1$ and $\alpha_2 T = \beta_2$. Then $\sigma_T \in \text{Aut}(Sp(2\nu, q))$ such that $\sigma_T([\alpha_1]) = [\beta_1]$ and $\sigma_T([\alpha_2]) = [\beta_2]$. Hence $Sp(2\nu, q)$ is edge transitive. \hfill \Box

When $q = 2$, we have the following

Proposition 3.3. $\text{Aut}(Sp(2\nu, 2)) \cong Sp_{2\nu}(\mathbb{F}_2)$.

Proof. Let

$$
\sigma : \ Sp_{2\nu}(\mathbb{F}_2) \rightarrow \text{Aut}(Sp(2\nu, 2))
T \mapsto \sigma_T.
$$

Then, by 3.1, $\sigma$ is an injection. Clearly, $\sigma$ preserves the operation. It remains to show that, for any $\tau \in \text{Aut}(Sp(2\nu, 2))$, there exists a $T \in Sp_{2\nu}(\mathbb{F}_2)$ such that $\tau = \sigma_T$.

Note that, for any $\alpha \neq 0 \in \mathbb{F}_2^{(2\nu)}$, we have that $[\alpha] = \{0, \alpha\}$. We will denote the uniquely defined element $\tau([\alpha]) \setminus \{0\}$ by $\tau(\alpha)$ and set $\tau(0) = 0$. Then from $\tau \in \text{Aut}(Sp(2\nu, 2))$ we see that $\alpha K' \beta = \tau(\alpha) K' \tau(\beta)$ for any $\alpha, \beta \in \mathbb{F}_2^{(2\nu)}$ (not necessarily non-zero). Fix any $\alpha \in \mathbb{F}_2^{(2\nu)}$. Let $\beta_1, \beta_2 \in \mathbb{F}_2^{(2\nu)}$. Then

$$
\begin{align*}
\alpha K' \beta_1 &= \tau(\alpha) K' \tau(\beta_1), \\
\alpha K' \beta_2 &= \tau(\alpha) K' \tau(\beta_2).
\end{align*}
$$

Thus

$$
\alpha K' (\beta_1 + \beta_2) = \tau(\alpha) K' (\tau(\beta_1) + \tau(\beta_2)).
$$

But

$$
\alpha K' (\beta_1 + \beta_2) = \tau(\alpha) K' (\tau(\beta_1 + \beta_2)),
$$

hence

$$
\tau(\alpha) K' (\tau(\beta_1 + \beta_2) + \tau(\beta_1) + \tau(\beta_2)) = 0.
$$

This is true for any $\alpha \in \mathbb{F}_2^{(2\nu)}$, it follows that $\tau(\beta_1 + \beta_2) + \tau(\beta_1) + \tau(\beta_2) = 0$, i.e., $\tau(\beta_1 + \beta_2) = \tau(\beta_1) + \tau(\beta_2)$. Set

$$
T = \begin{pmatrix}
\tau(1, 0, \ldots, 0) \\
\tau(0, 1, \ldots, 0) \\
\vdots \\
\tau(0, 0, \ldots, 1)
\end{pmatrix}.
$$

Then $\tau(\alpha) = \alpha T$ for any $\alpha \in \mathbb{F}_2^{(2\nu)}$. Thus $T$ is nonsingular. By 3.1 $T \in Sp_{2\nu}(\mathbb{F}_2)$ and $\tau = \sigma_T$ as required. \hfill \Box

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From now on, we assume that $q > 2$. In $\mathbb{F}_q^{(2\nu)}$, let us set

\[
\begin{align*}
e_1 &= (1, 0, 0, 0, \ldots, 0, 0), \\
f_1 &= (0, 1, 0, 0, \ldots, 0, 0), \\
e_2 &= (0, 0, 1, 0, \ldots, 0, 0), \\
f_2 &= (0, 0, 0, 1, \ldots, 0, 0), \\
&\vdots \\
e_\nu &= (0, 0, 0, 0, \ldots, 1, 0), \\
f_\nu &= (0, 0, 0, 0, \ldots, 0, 1).
\end{align*}
\]

Then $e_i, f_i, i = 1, \ldots, \nu$, form a basis of $\mathbb{F}_q^{(2\nu)}$ and $e_iK'f_i = 1, e_iK'e_j = 0, f_iK'f_j = 0, i, j = 1, \ldots, \nu$, and $e_iK'f_j = 0, i \neq j, i, j = 1, \ldots, \nu$.

In order to describe $\text{Aut}(Sp(2\nu, q))$ for any prime power $q$, we need some definition from group theory. Let $\varphi$ be the natural action of $\text{Aut}(\mathbb{F}_q)$ on the group $\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*$ ($\nu$ in number) defined by

\[
\varphi(\pi)((k_1, \ldots, k_\nu)) = (\pi(k_1), \ldots, \pi(k_\nu)), \text{ for all } \pi \in \text{Aut}(\mathbb{F}_q) \text{ and } k_1, \ldots, k_\nu \in \mathbb{F}_q^*.
\]

then the semi-direct product of $\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*$ by $\text{Aut}(\mathbb{F}_q)$, corresponding to $\varphi$, denoted by $(\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*) \rtimes \varphi \text{ Aut}(\mathbb{F}_q)$, is the group consisting of all elements of the form $(k_1, \ldots, k_\nu, \pi)$, where $k_1, \ldots, k_\nu \in \mathbb{F}_q^*$ and $\pi \in \text{Aut}(\mathbb{F}_q)$, with multiplication defined by

\[
(k_1, \ldots, k_\nu, \pi)(k'_1, \ldots, k'_\nu, \pi') = (k_1 \pi(k'_1), \ldots, k_\nu \pi(k'_\nu), \pi \pi').
\]

Then the main result about $\text{Aut}(Sp(2\nu, q))$ is as follows.

**Theorem 3.4.** Regard $PSp_{2\nu}(\mathbb{F}_q)$ as a subgroup of $\text{Aut}(Sp(2\nu, q))$ and let $E$ be the subgroup of $\text{Aut}(Sp(2\nu, q))$ defined as follows

\[
E = \{ \sigma \in \text{Aut}(Sp(2\nu, q)) : \sigma([e_i]) = [e_i], \sigma([f_i]) = [f_i], i = 1, \ldots, \nu \}.
\]

Then

1. $\text{Aut}(Sp(2\nu, q)) = PSp_{2\nu}(\mathbb{F}_q) \cdot E$;
2. If $\nu = 1$, then $E$ is isomorphic to the symmetric group on $q-1$ elements;
3. If $\nu > 1$, then

\[
E \cong (\bigoplus_{\nu} \mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*) \rtimes \varphi \text{ Aut}(\mathbb{F}_q).
\]

**Proof.** (1) Let $\tau \in \text{Aut}(Sp(2\nu, q))$. Suppose that $\tau([e_i]) = [e'_i], \tau([f_i]) = [f'_i], i = 1, \ldots, \nu$. Then $e'_iK'f'_i \neq 0, e'_iK'e_j = 0, f'_iK'f'_j = 0, i, j = 1, \ldots, \nu$ and $e'_iK'f'_j = 0, i \neq j, i, j = 1, \ldots, \nu$. We may choose $e'_i, f'_i, i = 1, \ldots, \nu$, such that $e'_iK'f'_i = 1, i = 1, \ldots, \nu$. Then
Then $AK' \cdot A = K = A'K' \cdot A'$. Thus, by [6, Lemma 3.11], there exists $T \in Sp_{2\nu}(\mathbb{F}_q)$ such that $A = A' \cdot T$, i.e., $e_i^T = e_i$, $f_i^T = f_i$, $i = 1, \ldots, \nu$. Set $\tau_1 = \sigma T \tau$. Then $\tau_1([e_i]) = [e_i]$, $\tau_1([f_i]) = [f_i]$, $i = 1, \ldots, \nu$, hence $\tau_1 \in E$. Thus $\tau \in PSp_{2\nu}(\mathbb{F}_q) : E$. It follows that $\text{Aut}(Sp(2\nu, q)) = PSp_{2\nu}(\mathbb{F}_q) : E$.

(2) When $\nu = 1$, it is clear that $E$ is isomorphic to the symmetric group on the $q - 1$ vertices of $Sp(2, q)$ since $Sp(2, q)$ is a complete graph.

(3) Suppose that $\nu > 1$. Firstly, let us write out some elements of $E$. Let $k_1, \ldots, k_\nu \in \mathbb{F}_q^*$ and $\pi \in \text{Aut}(\mathbb{F}_q)$. Let $\sigma(k_1, \ldots, k_\nu, \pi)$ be the map which takes any vertex $[a_1, a_2, a_3, a_4, \ldots, a_{2\nu-1}, a_{2\nu}]$ of $Sp(2\nu, q)$ to the vertex

$$[\pi(a_1), k_1 \pi(a_2), k_2 \pi(a_3), k_1 k_2^{-1} \pi(a_4), \ldots, k_\nu \pi(a_{2\nu-1}), k_1 k_\nu^{-1} \pi(a_{2\nu})].$$

Then it is clear that $\sigma(k_1, \ldots, k_\nu, \pi)$ is well-defined. Furthermore, it is easy to see that $\sigma(k_1, \ldots, k_\nu, \pi)$ is injective, but the vertex set of $Sp(2\nu, q)$ is finite, $\sigma(k_1, \ldots, k_\nu, \pi)$ is a bijection from $V(\text{Sp}(2\nu, q))$ to itself. Let $\alpha = [a_1, a_2, a_3, a_4, \ldots, a_{2\nu-1}, a_{2\nu}]$, $\beta = [a'_1, a'_2, a'_3, a'_4, \ldots, a'_{2\nu-1}, a'_{2\nu}]$ be two vertices of $Sp(2\nu, q)$. If $\alpha \not\sim \beta$, then, by definition,

$$(a_1 a'_2 - a_2 a'_1) + (a_3 a'_4 - a_4 a'_3) + \ldots + (a_{2\nu-1} a'_{2\nu} - a_{2\nu} a'_{2\nu-1}) = 0,$$

which implies that

$$\sigma(k_1, \ldots, k_\nu, \pi)(\alpha) \not\sim \sigma(k_1, \ldots, k_\nu, \pi)(\beta).$$

Since the edges set of $Sp(2\nu, q)$ is finite, $\sigma(k_1, \ldots, k_\nu, \pi)$ is a bijection from $V(\text{Sp}(2\nu, q))$ to itself. Let $\alpha = [a_1, a_2, a_3, a_4, \ldots, a_{2\nu-1}, a_{2\nu}]$, $\beta = [a'_1, a'_2, a'_3, a'_4, \ldots, a'_{2\nu-1}, a'_{2\nu}]$ be two vertices of $Sp(2\nu, q)$. If $\alpha \not\sim \beta$, then, by definition,

$$\sigma(k_1, \ldots, k_\nu, \pi)([e_i]) = [e_i], \sigma(k_1, \ldots, k_\nu, \pi)([f_i]) = [f_i], i = 1, \ldots, \nu,$$

hence, $\sigma(k_1, \ldots, k_\nu, \pi) \in E$.

If we define a map $h$ as $(k_1, \ldots, k_\nu, \pi) \mapsto \sigma(k_1, \ldots, k_\nu, \pi)$, then it is easy to verify that $h$ is a group homomorphism from $(\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*) \times \text{Aut}(\mathbb{F}_q)$ to $E$. It is also easy to see that if $(k_1, \ldots, k_\nu, \pi) \neq (k'_1, \ldots, k'_\nu, \pi')$ then $\sigma(k_1, \ldots, k_\nu, \pi) \neq \sigma(k'_1, \ldots, k'_\nu, \pi')$. Thus, to show that $h$ is a group isomorphism, it remains to show that every element of $E$ is of the form $\sigma(k_1, \ldots, k_\nu, \pi)$.

Suppose that $\sigma \in E$. Note that if $\sigma([a_1, a_2, \ldots, a_{2\nu}]) = [b_1, b_2, \ldots, b_{2\nu}]$, then $a_{2\nu-1} \neq 0$ if and only if $[a_1, a_2, \ldots, a_{2\nu}] \sim [f_i]$ and $a_{2\nu} \neq 0$ if and only if $[a_1, a_2, \ldots, a_{2\nu}] \sim [e_i]$, and similar results are also true for $b_i$. But $\sigma([e_i]) = [e_i]$ and $\sigma([f_i]) = [f_i]$, it follows that $a_i = 0$ if and only if $b_i = 0$. For any vertex $[a_1, a_2, \ldots, a_{2\nu}]$, if $a_1 = \cdots = a_{i-1} = 0$ and $a_i \neq 0$ then $[a_1, a_2, \ldots, a_{2\nu}]$ can be uniquely written as $[0, \ldots, 0, 1, a_{i+1}, \ldots, a_{2\nu}]$ and $\sigma([a_1, a_2, \ldots, a_{2\nu}])$ can be uniquely written as
Let us show how to determine \( b'_{i+1}, \ldots, b'_{2\nu} \) from \( a'_{i+1}, \ldots, a'_{2\nu} \). We will use frequently the fact that, for any vertices \( [\alpha], [\beta] \), if \( [\alpha] \not\triangleright [\beta] \) then \( \sigma([\alpha]) \not\triangleright \sigma([\beta]) \).

In the following, we will denote \([a_1, a'_1, a_2, a'_2, \ldots, a_{\nu}, a'_{\nu}]\) by \( \sum_{i=1}^\nu a_i[e_i] + \sum_{i=1}^\nu a'_i[f_i] \), for example, \([a, 0, 0, \ldots, 0] \) is denoted by \( a(e_1) + b[f_1] \). Since \( \sigma \) is a bijection from \( V(Sp(2\nu, q)) \) to itself, we have permutations \( \pi_i, i = 2, \ldots, 2\nu, \) of \( F_q \) with \( \pi(0) = 0 \) such that

\[
\sigma([e_1] + a_{2i-1}[e_i]) = [e_1] + \pi_{2i-1}(a_{2i-1})[e_i] \\
\sigma([e_1] + a_{2i}[f_i]) = [e_1] + \pi_{2i}(a_{2i})[f_i].
\]

We firstly consider the cases \( \sigma([0, 1, a_3, \ldots, a_{2\nu}] \) and \( \sigma([1, a_2, a_3, \ldots, a_{2\nu}] \). Let \( \sigma([0, 1, a_3, \ldots, a_{2\nu}] = [0, 1, a'_3, \ldots, a'_{2\nu}] \) and \( j \geq 1 \). If \( a_{2j+1} \neq 0 \), then, from \( [0, 1, a_3, \ldots, a_{2\nu}] \not\triangleright [e_1] + a_{2j-1}[f_{j+1}] \) we have \( [0, 1, a'_3, \ldots, a'_{2\nu}] \not\triangleright [e_1] + \pi_{2j+2}(a_{2j+1})[f_{j+1}] \), hence, \( a'_{2j+1} = \pi_{2j+2}(a_{2j+1})^{-1} \). If \( a_{2j+2} \neq 0 \), then from \( [0, 1, a_3, \ldots, a_{2\nu}] \not\triangleright [e_1] - a_{2j+2}[e_{j+1}] \) we have \( [0, 1, a'_3, \ldots, a'_{2\nu}] \not\triangleright [e_1] + \pi_{2j+1}(-a_{2j+2})[e_{j+1}] \), hence, \( a'_{2j+2} = -\pi_{2j+1}(-a_{2j+2})^{-1} \). Thus

\[
\sigma([0, 1, a_3, \ldots, a_{2\nu}] = [0, 1, a'_3, \ldots, a'_{2\nu}],
\]

where \( a'_{2j+1} = \pi_{2j+2}(a_{2j+1})^{-1} \) if \( a_{2j+1} \neq 0 \) and \( a'_{2j+2} = -\pi_{2j+1}(-a_{2j+2})^{-1} \) if \( a_{2j+2} \neq 0 \).

For the case \( \sigma([1, a_2, a_3, \ldots, a_{2\nu}] \). Let \( \sigma([1, a_2, a_3, \ldots, a_{2\nu}] = [1, a'_2, a'_3, \ldots, a'_{2\nu}] \). From \( [1, a_2, a_3, \ldots, a_{2\nu}] \not\triangleright [e_1] + a_2[f_1] \) we get \( [1, a'_2, a'_3, \ldots, a'_{2\nu}] \not\triangleright [e_1] + \pi_2(a_2)[f_1] \), hence, \( a'_2 = \pi_2(a_2) \). Let \( j \geq 1 \). If \( a_{2j+1} \neq 0 \), then, from \( [1, a_2, a_3, \ldots, a_{2\nu}] \not\triangleright [f_i] - a_{2j+1}[f_{j+1}] \) and \( \sigma([f_i] - a_{2j+1}[f_{j+1}] = [f_i] - \pi_{2j+1}(a_{2j+1})^{-1}[f_{j+1}] \) as been shown above, we have \( [1, a'_2, a'_3, \ldots, a'_{2\nu}] \not\triangleright [f_i] - \pi_{2j+1}(a_{2j+1})^{-1}[f_{j+1}] \), hence, \( a''_{2j+1} = \pi_{2j+1}(a_{2j+1}) \). Similarly, if \( a_{2j+2} \neq 0 \), then from \( [1, a_2, a_3, \ldots, a_{2\nu}] \not\triangleright [f_i] + a_{2j+2}[e_{j+1}] \) we have \( [1, a'_2, a'_3, \ldots, a'_{2\nu}] \not\triangleright [f_i] + \pi_{2j+2}(a_{2j+2})^{-1}[e_{j+1}] \), hence, \( a''_{2j+2} = \pi_{2j+2}(a_{2j+2}) \). Thus, for any \( a_2, a_3, \ldots, a_{2\nu} \in F_q \),

\[
\sigma([1, a_2, a_3, \ldots, a_{2\nu}] = [1, \pi_2(a_2), \pi_3(a_3), \ldots, \pi_\nu(a_{2\nu})].
\]

Then, let \( i \geq 2 \), we discuss the general cases \( \sigma([0, 0, 0, 1, a_{2i+1}, \ldots, a_{2\nu}] \) and \( \sigma([0, 0, 0, 1, a_{2i+1}, \ldots, a_{2\nu}] \). The above results of case \( i = 1 \) will be used. Let \( \sigma([0, 0, 0, 1, a_{2i+1}, \ldots, a_{2\nu}] = [0, 0, 0, 1, a'_{2i+1}, \ldots, a'_{2\nu}] \) and \( j \geq i \). If \( a_{2j+1} \neq 0 \), then, from

\[
[0, 0, 0, 1, a'_{2i+1}, \ldots, a'_{2\nu}] \not\triangleright [e_i] + a_{2j+1}[f_{j+1}] \\
\text{and} \sigma([e_i] + a_{2j+1}[f_{j+1}] = [e_i] + \pi_{2i-1}(1)[e_i] + \pi_{2j+2}(a_{2j+1})^{-1}[f_{j+1}] \text{ as been shown above, we have}
\]

\[
[0, 0, 0, 1, a'_{2i+1}, \ldots, a'_{2\nu}] \not\triangleright [e_i] + \pi_{2i-1}(1)[e_i] + \pi_{2j+2}(a_{2j+1})^{-1}[f_{j+1}],
\]

hence, \( a'_{2j+1} = \pi_{2i-1}(1)\pi_{2j+2}(a_{2j+1})^{-1} \). Similarly, if \( a_{2j+2} \neq 0 \), then from

\[
[0, 0, 0, 1, a'_{2i+1}, \ldots, a'_{2\nu}] \not\triangleright [e_i] + a_{2j+2}[e_{j+1}]
\]

we have

\[
[0, 0, 0, 1, a'_{2i+1}, \ldots, a'_{2\nu}] \not\triangleright [e_i] + \pi_{2i-1}(1)[e_i] + \pi_{2j+1}(-a_{2j+2})^{-1}[e_{j+1}],
\]
hence, \( a'_{2j+2} = -\pi_{2i-1}(1)\pi_{2j+1}(-a'_{2j+2})^{-1} \). Thus,

\[
\sigma([0, \ldots, 0, 1, a_{2i+1}, \ldots, a_{2\nu}]) = [0, \ldots, 0, 1, a'_{2i+1}, \ldots, a'_{2\nu}],
\]

where \( a'_{2j+1} = \pi_{2i-1}(1)\pi_{2j+2}(a'_{2j+1})^{-1} \) if \( a_{2j+1} \neq 0 \) and \( a'_{2j+2} = -\pi_{2i-1}(1)\pi_{2j+1}(-a'_{2j+2})^{-1} \) if \( a_{2j+2} \neq 0 \).

Finally, for the case \( \sigma([0, \ldots, 0, 1, a_{2i}, \ldots, a_{2\nu}]) \). Let \( \sigma([0, \ldots, 0, 1, a_{2i}, \ldots, a_{2\nu}]) = [0, \ldots, 0, 1, a''_{2i}, \ldots, a''_{2\nu}] \). From

\[
[0, \ldots, 0, 1, a_{2i}, \ldots, a_{2\nu}] \not\sim [e_1] + [e_i] + a_{2i}[f_i]
\]

we get

\[
[0, \ldots, 0, 1, a''_{2i}, \ldots, a''_{2\nu}] \not\sim [e_1] + \pi_{2i-1}(1)[e_i] + \pi_{2i}(a_{2i})[f_i],
\]

hence, \( a''_{2i} = \pi_{2i-1}(1)^{-1}\pi_{2i}(a_{2i}) \). Let \( j \geq i \). If \( a_{2j+1} \neq 0 \), then from

\[
[0, \ldots, 0, 1, a_{2i}, \ldots, a_{2\nu}] \not\sim [f_i] - a_{2j+1}[f_{j+1}]
\]

we have

\[
[0, \ldots, 0, 1, a''_{2i}, \ldots, a''_{2\nu}] \not\sim [f_i] - \pi_{2i-1}(1)^{-1}\pi_{2j+1}(a_{2j+1})^{-1}[f_{j+1}],
\]

hence, \( a''_{2j+1} = \pi_{2i-1}(1)^{-1}\pi_{2j+1}(a_{2j+1}) \). If \( a_{2j+2} \neq 0 \), then from

\[
[0, \ldots, 0, 1, a_{2i}, \ldots, a_{2\nu}] \not\sim [f_i] + a_{2j+2}[e_{j+1}]
\]

we have

\[
[0, \ldots, 0, 1, a''_{2i}, \ldots, a''_{2\nu}] \not\sim [f_i] + \pi_{2i-1}(1)^{-1}\pi_{2j+2}(a_{2j+2})^{-1}[e_{j+1}],
\]

hence, \( a''_{2j+2} = \pi_{2i-1}(1)^{-1}\pi_{2j+2}(a_{2j+2}) \). Thus, for any \( a_{2i}, a_{2i+1}, \ldots, a_{2\nu} \in \mathbb{F}_q \),

\[
\sigma([0, \ldots, 0, 1, a_{2i}, a_{2i+1}, \ldots, a_{2\nu}]) = [0, \ldots, 0, 1, \pi_{2i-1}(1)^{-1}\pi_{2i}(a_{2i}), \pi_{2i-1}(1)^{-1}\pi_{2i+1}(a_{2i+1}), \ldots, \pi_{2i-1}(1)^{-1}\pi_{2\nu}(a_{2\nu})].
\]

Having represented \( \sigma \) by \( \pi_i, i = 2, \ldots, 2\nu \), let us discuss some properties of \( \pi_i \).

**Lemma 3.5.**

1. For any \( i \geq 1 \) and \( a \in \mathbb{F}_q \),

\[
\pi_{2i+1}(1)\pi_{2i+2}(a) = \pi_{2i+2}(1)\pi_{2i+1}(a) = \pi_2(a);
\]

2. For any \( i \geq 2 \) and \( a, b \in \mathbb{F}_q \),

\[
\begin{align*}
\pi_i(a + b) &= \pi_i(a) + \pi_i(b); \\
\pi_i(-a) &= -\pi_i(a); \\
\pi_i(ab) &= \pi_i(a)\pi_i(b)\pi_i(1)^{-1}; \\
\pi_i(a^{-1}) &= \pi_i(a)^{-1}\pi_i(1)^2 \text{ if } a \neq 0.
\end{align*}
\]

**Proof.** (1) We may assume that \( a \neq 0 \). Since \([e_1] + a[e_{i+1}] + a[f_{i+1}] \not\sim [e_{i+1}] + [f_{i+1}]\), it follows that \( \sigma([e_1] + a[e_{i+1}] + a[f_{i+1}]) \not\sim \sigma([e_{i+1}] + [f_{i+1}]) \), but

\[
\sigma([e_1] + a[e_{i+1}] + a[f_{i+1}]) = [e_1] + \pi_{2i+1}(a)[e_{i+1}] + \pi_{2i+2}(a)[f_{i+1}],
\]

\[
\sigma([e_{i+1}] + [f_{i+1}]) = [e_{i+1}] + \pi_{2i+1}(1)^{-1}\pi_{2i+2}(1)[f_{i+1}],
\]

we have that

\[
\pi_{2i+1}(1)^{-1}\pi_{2i+2}(1)\pi_{2i+1}(a) - \pi_{2i+2}(a) = 0,
\]
\[ \pi_{2i+1}(1) \pi_{2i+2}(a) = \pi_{2i+2}(1) \pi_{2i+1}(a). \]

Similarly, since \([e_1] + a[f_1] + [e_{i+1}] \neq [e_1] + a[f_{i+1}],\) we have that \([e_1] + \pi_2(a)[f_1] + \pi_{2i+1}(1)[e_{i+1}] \neq [e_1] + \pi_{2i+2}(a)[f_{i+1}],\) hence, \(\pi_{2i+1}(1) \pi_{2i+2}(a) = \pi_2(a).\)

(2) From \([e_1] + (a + b)[f_1] + [e_2] \neq [e_1] + a[f_1] + b[f_2]\) we have that

\[ [e_1] + \pi_2(a + b)[f_1] + \pi_3(1)[e_2] \neq [e_1] + \pi_2(a)[f_1] + \pi_4(b)[f_2]. \]

Then \(\pi_2(a) - \pi_2(a + b) + \pi_3(1)\pi_4(b) = 0,\) but \(\pi_3(1)\pi_4(b) = \pi_2(b),\) hence, \(\pi_2(a + b) = \pi_2(a) + \pi_2(b).\) It turns out from (1) that this equality holds for all \(i \geq 2.\) Thus \(\pi_i(-a) = -\pi_i(a)\) as \(\pi_i(0) = 0.\)

For multiplication, let \(i \geq 1,\) from \([e_1] + b[e_{i+1}] + ab[f_{i+1}] \not\sim [e_{i+1}] + a[f_{i+1}]\) we get that

\[ [e_1] + \pi_{2i+1}(b)[e_{i+1}] + \pi_{2i+2}(ab)[f_{i+1}] \not\sim [e_{i+1}] + \pi_{2i+1}(1)^{-1}\pi_{2i+2}(a)[f_{i+1}], \]

hence, \(\pi_{2i+1}(b)\pi_{2i+1}(1)^{-1}\pi_{2i+2}(a) - \pi_{2i+2}(ab) = 0,\) but \(\pi_{2i+1}(b)\pi_{2i+1}(1)^{-1} = \pi_{2i+2}(b)\pi_{2i+2}(1)^{-1}.\)

Thus

\[ \pi_{2i+2}(ab) = \pi_{2i+2}(a)\pi_{2i+2}(b)\pi_{2i+2}(1)^{-1}. \]

It follows from \(\pi_{2i+1}(1)\pi_{2i+2}(a) = \pi_{2i+2}(1)\pi_{2i+1}(a)\) and \(\pi_{2i+1}(1)\pi_{2i+2}(1) = \pi_{2i+2}(1)\pi_{2i+1}(1)\) that the above equality also holds for \(2i + 1.\) It remains to consider \(\pi_2.\) We have

\[ \pi_2(ab) = \pi_3(1)\pi_4(ab) = \pi_3(1)\pi_4(1)^{-1}\pi_4(a)\pi_4(b) = \pi_3(1)^{-1}\pi_4(1)^{-1}\pi_2(a)\pi_2(b) = \pi_2(a)\pi_2(b)\pi_2(1)^{-1}. \]

Finally, if \(a \neq 0,\) then from \(\pi_i(1) = \pi_i(aa^{-1}) = \pi_i(a)\pi_i(a^{-1})\pi_i(1)^{-1}\) we obtain that \(\pi_i(a^{-1}) = \pi_i(a)^{-1}\pi_i(1)^2,\) then the proof of lemma is complete.

We continue the proof of the theorem. Let us denote the identity automorphism on \(\mathbb{F}_q\) by \(\pi_1.\) Then when \(i = 1,\) (3) reduces to (1) and (4) reduces to (2). Therefore (3) and (4) hold for all \(i,\) where \(1 \leq i \leq \nu.\) By the above lemma, for any \(i \geq 1,\) we can rewrite (3) in the form of (4) as follows. In (3), for any \(j \geq i,\) we have

\[ a_{2j+1}' = \pi_{2i-1}(1)\pi_{2j+2}(a_{2j+1})^{-1} = \pi_{2i-1}(1)\pi_{2j+2}(a_{2j+1})\pi_{2j+2}(1)^{-2} = \pi_{2i-1}(1)\pi_{2j+1}(1)^{-1}\pi_{2j+2}(1)^{-1}\pi_{2j+1}(1) = \pi_{2i-1}(1)\pi_2(1)^{-1}\pi_{2j+1}(1) = \pi_2(1)^{-1}\pi_{2j+1}(a_{2j+1}), \]

\[ a_{2j+1} = \pi_{2i-1}(1)\pi_{2j+2}(a_{2j+1})^{-1} = \pi_{2i-1}(1)\pi_{2j+2}(a_{2j+1})\pi_{2j+2}(1)^{-2} = \pi_{2i-1}(1)\pi_{2j+1}(1)^{-1}\pi_{2j+2}(1)^{-1}\pi_{2j+1}(1) = \pi_{2i-1}(1)\pi_2(1)^{-1}\pi_{2j+1}(1) = \pi_2(1)^{-1}\pi_{2j+1}(a_{2j+1}), \]

\[ 10 \]
and
\[
a'_{2j+2} = -\pi_{2i-1}(1)\pi_{2j+1}(a_{2j+2})^{-1}
= \pi_{2i-1}(1)\pi_{2j+1}(a_{2j+2})^{-1}
= \pi_{2i-1}(1)\pi_{2j+1}(a_{2j+2})\pi_{2j+1}(1)^{-2}
= \pi_{2i-1}(1)\pi_{2j+1}(1)^{-1}\pi_{2j+2}(1)^{-1}\pi_{2j+2}(a_{2j+2})
= \pi_{2i-1}(1)\pi_{2}(1)^{-1}\pi_{2j+2}(a_{2j+2})
= \pi_{2}(1)^{-1}\pi_{2j+2}(a_{2j+2}).
\]
Hence, for any \(a_{2i+1}, \ldots, a_{2\nu} \in \mathbb{F}_q\),
\[(5) \quad \sigma([0, \ldots, 0, 1, a_{2i+1}, \ldots, a_{2\nu}]) = [0, \ldots, 0, 1, \pi_{2i}(1)^{-1}\pi_{2i+1}(a_{2i+1}), \ldots, \pi_{2}\pi(1)^{-1}\pi_{2\nu}(a_{2\nu})],\]
which is of the same form as (4).

Now let \(k_1 = \pi_{2}(1), \pi = k_1^{-1}\pi_2, k_2 = \pi_3(1), k_3 = \pi_5(1), \ldots, k_{\nu} = \pi_{2\nu-1}(1)\). Then \(\pi \in \text{Aut}(\mathbb{F}_q)\), \(\pi_2 = k_1\pi, \pi_3 = k_2\pi, \pi_4 = k_1k_2^{-1}\pi, \ldots, \pi_{2\nu-1} = k_{\nu}\pi, \pi_{2\nu} = k_1k_{\nu}^{-1}\pi\). Assembling (4) and (5), we obtain
\[
\sigma([a_1, a_2, a_3, a_4, \ldots, a_{2\nu-1}, a_{2\nu}])
= [\pi(a_1), k_1\pi(a_2), k_2\pi(a_3), k_1k_2^{-1}\pi(a_4), \ldots, k_{\nu}\pi(a_{2\nu-1}), k_1k_{\nu}^{-1}\pi(a_{2\nu})].
\]
Hence \(\sigma = h(k_1, \ldots, k_{\nu}, \pi)\), as required.

**Corollary 3.6.** When \(\nu = 1\),
\[
|\text{Aut}(Sp(2, q))| = q(q^2 - 1) \cdot (q - 2)!,
\]
and when \(\nu \geq 2\),
\[
|\text{Aut}(Sp(2\nu, q))| = q^{\nu^2} \prod_{i=1}^{\nu}(q^{2i} - 1) \cdot [\mathbb{F}_q : \mathbb{F}_p].
\]

**Proof.** Note that \(PSp_{2\nu}(\mathbb{F}_q) \cap E\) consists of \(\sigma\) which is reduced from some matrix of the form \(\text{diag}(k_1l_1, k_2l_2, \ldots, k_{\nu}l_{\nu})\), with \(k_il_i = 1, i = 1, \ldots, \nu\). Thus \(|PSp_{2\nu}(\mathbb{F}_q) \cap E| = \frac{1}{2}(q - 1)^{\nu}\). Hence
\[
|\text{Aut}(Sp(2\nu, q))| = \frac{|PSp_{2\nu}(\mathbb{F}_q)||E|}{|PSp_{2\nu}(\mathbb{F}_q) \cap E|}
= \frac{\frac{1}{2}q^{\nu^2} \prod_{i=1}^{\nu}(q^{2i} - 1) \cdot |E|}{\frac{1}{2}(q - 1)^{\nu}}.
\]
Thus, when $\nu = 1$, $|\text{Aut}(Sp(2, q))| = q(q^2 - 1) \cdot (q - 2)!$, and when $\nu \geq 2$,
\[
|\text{Aut}(Sp(2\nu, q))| = \frac{\frac{1}{2}q^{\nu^2} \prod_{i=1}^{\nu} (q^{2i} - 1) \cdot (q - 1)^{\nu} \cdot |\text{Aut}(\mathbb{F}_q)|}{\frac{1}{2}(q - 1)^\nu} = q^{\nu^2} \prod_{i=1}^{\nu} (q^{2i} - 1) \cdot |\text{Aut}(\mathbb{F}_q)| = q^{\nu^2} \prod_{i=1}^{\nu} (q^{2i} - 1) \cdot [\mathbb{F}_q : \mathbb{F}_p],
\]
as is well-known that $|\text{Aut}(\mathbb{F}_q)| = [\mathbb{F}_q : \mathbb{F}_p]$ where $p = \text{char}(\mathbb{F}_q)$.

\begin{thebibliography}{9}


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