ON THE INTEGRITY OF GRAPHS

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ABSTRACT
The integrity of a graph \( G = (V, E) \) is defined as \( I(G) = \min\{|S| + m(G - S) : S \subseteq V(G)\} \), where \( m(G - X) \) denotes the order of the largest component in the graph \( G - X \). This is a better parameter to measure the stability of a network, as it takes into account both the amount of work done to damage the network and how badly the network is damaged. In this paper, the maximum networks are obtained with prescribed order and integrity, and a method for constructing this sort of networks is also presented. Finally, we give the trees of minimum integrity with given order and maximum degree.

KEY WORDS
Integrity, Maximum network, Nonlinear integer programming, Saturated tree.

1 Introduction

In an analysis of the vulnerability of a communication network to disruption, two qualities that come to mind are the number of elements that are not functioning and the size of the largest remaining subnetwork within which mutual communication can still occur. In particular, in an adversarial relationship, it would be desirable for an opponent’s network to be such that the two qualities can be made to be simultaneously small.

The integrity of a graph \( G = (V, E) \), which was introduced in [1] as a useful measure of the vulnerability of the graph, is defined as follows:

\[
I(G) = \min\{|S| + m(G - S) : S \subseteq V(G)\},
\]

where \( m(G - S) \) denotes the order of the largest component in \( G - S \).

Unlike the connectivity measures, integrity shows not only the difficulty to break down the network but also the damage that has been caused. In [1] Barefoot et al gave some basic results on integrity and Clark et al [4] proved that the determination of the integrity is NP-complete. In [5] Moazzami et al compared the integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. To know more about integrity, one can see a survey of integrity in [2].

A vertex subset \( S \) of a graph \( G \) is called an \( I \)-set of \( G \) if it satisfies that \( I(G) = |S| + m(G - S) \).

A network is called the maximum (minimum) network if it has maximum (minimum) number of edges with prescribed order and some properties.

As a useful parameter to measure the stability of networks, what we are interested in it is the following question: For any given two integers \( n \) and \( I \) such that \( 2 \leq I \leq n \), how can we construct a network with order \( n \) and integrity \( I \)? The paper is organized as follows: In Section 2, the maximum network with prescribed order and integrity is given, and a method for constructing such maximum network is presented in Section 3. In Section 4, we give the minimum integrity of trees with given order and maximum degree.

Throughout this paper, a graph \( G = (V, E) \) always means a simple connected graph with vertex set \( V \) and edge set \( E \). We use Bondy and Murty [3] for terminology and notations not defined here. \( E_n \) and \( K_n \), respectively, denotes the null graph and complete graph of order \( n \). If \( S \) is a nonempty subset of \( V \), we use \( G[S] \) denotes the induced subgraph of \( G \). We shall use \( |x| \) for the largest integer not larger than a real number \( x \). \( \Delta \) denotes the maximum degree of a graph. A \( \Delta \)-edge is an edge which joins two vertices of degree \( \Delta \). A leaf means a vertex of degree 1. An edge incident with a leaf is called a leaf-edge. An edge is said to be subdivided when it is replaced by a path of length two connecting its ends, and the internal vertex in this path is a new vertex.

2 The maximum network with prescribed order and integrity

In this section, we give the maximum network with given order and integrity. The following lemma is used in the proof of our main theorem.

Lemma 2.1 ([2]) If \( G \) is a connected graph with order \( n \) and integrity \( I \), then \( 2 \leq I(G) \leq n \), and \( I(G) = n \) if and
Suppose that the components of

\[ \text{Proof.} \]

So, by

\[ \frac{1}{2} I(I - 1) + (I - 1)(n - I). \]

**Theorem 2.1**

\[ \max_{G \in G[n, I]} |E(G)| = \frac{1}{2} I(I - 1) + (I - 1)(n - I). \]

**Proof.** Let \( S \subseteq V(G) \) be an \( I \)-set of \( G \), i.e.,

\[ I(G) = |S| + m(G - S). \]

Suppose that the components of \( G - S \) are \( G_1, G_2, \ldots, G_p \), and let \( |S| = x, |V(G_i)| = n_i \) (\( i = 1, 2, \ldots, p \)). Then we have

\[ G - S = G_1 \cup G_2 \cup \cdots \cup G_p \]

and

\[ I(G) = x + m(G - S), \sum_{i=1}^{p} n_i = n - x. \]

If we want the number of edges of a graph \( G \), \( |E(G)| \) to achieve the maximum, the following statements must hold:

1. \( G[S] \) is a complete subgraph of \( G \),
2. All \( G_i \) (\( i = 1, 2, \ldots, p \)) are complete subgraphs of \( G \),
3. All vertices in \( S \) must be adjacent to all vertices in \( G_i \) (\( i = 1, 2, \ldots, p \)).

If the above three conditions are satisfied, let \( f(n_1, n_2, \ldots, n_p, x) \) denotes the number of edges of graph \( G \), then we have,

\[ f(n_1, n_2, \ldots, n_p, x) \]

\[ = \sum_{i=1}^{p} n_i - \frac{1}{2} + \left( \frac{x}{2} \right) + x \sum_{i=1}^{p} n_i \]

\[ = \frac{1}{2} \sum_{i=1}^{p} n_i^2 - \frac{1}{2} \sum_{i=1}^{p} n_i + \left( \frac{x}{2} \right) + x \sum_{i=1}^{p} n_i \]

\[ = \frac{1}{2} \left( \sum_{i=1}^{p} n_i \right)^2 + (x - \frac{1}{2}) \sum_{i=1}^{p} n_i + \left( \frac{x}{2} \right) - \sum_{1 \leq i < j \leq p} n_i n_j. \]

So, by \( \sum_{i=1}^{p} n_i = n - x \) we have

\[ f(n_1, n_2, \ldots, n_p, x) = \]

\[ \frac{1}{2} (n - x)^2 + (x - \frac{1}{2})(n - x) + \left( \frac{x}{2} \right) - \sum_{1 \leq i < j \leq p} n_i n_j. \]

To get the maximum value of \( f(n_1, n_2, \ldots, n_p, x) \), it is necessary to make

\[ \sum_{1 \leq i < j \leq p} n_i n_j \]

minimum. And it is easy to see that

\[ 1 \leq n_i \leq n - x - (p - 1) (i = 1, 2, \ldots, p). \]

Now let us determine the minimum value of

\[ \sum_{1 \leq i < j \leq p} n_i n_j, \]

i.e., solve the following nonlinear integer programming

\[ \min(N) = \sum_{1 \leq i < j \leq p} n_i n_j \]

s.t

\[ \left\{ \begin{array}{l}
1 \leq n_i \leq n - x - (p - 1) \\
i = 1, 2, \ldots, p \\
\sum_{i=1}^{p} n_i = n - x \\
n_i \in Z
\end{array} \right. \]

where \( N = (n_1, n_2, \ldots, n_p) \), \( Z \) is the set of positive integers.

To solve this problem, we first suppose that \( N^0 = (n_1^0, n_2^0, \ldots, n_p^0) \) is an arbitrary feasible solution of the above nonlinear integer programming. Let \( n_j^0 \) be the first number larger than \( 1 \) among \( n_1^0, n_2^0, \ldots, n_p^0 \), i.e.,

\[ N^0 = (1, 1, \ldots, 1, n_j^0, \ldots, n_p^0). \]

Construct a new feasible solution

\[ N^1 = (1, 1, \ldots, 1, n_{j+1}^0 + n_j^0 - 1, n_{j+2}^0, \ldots, n_p^0). \]

Then we have \( g(N^1) \leq g(N^0) \). Repeating the above process, we can finally get a feasible solution

\[ N' = (1, 1, \ldots, 1, n - x - p + 1). \]

Since \( N^0 \) is an arbitrary feasible solution, we know that \( N' \) is optimal. That’s to say, when \( n_1 = n_2 = \cdots = n_{p-1} = 1 \) and \( n_p = n - x - p + 1 \),

\[ \sum_{1 \leq i < j \leq p} n_i n_j \]
is minimized. Now substitute these values into 
\( f(n_1, n_2, \cdots, n_p, x) \), we have
\[
f(1, 1, \cdots, 1, n - x - p + 1, x) = \left( \frac{n - x - p + 1}{2} \right) + \left( \frac{x}{2} \right) + x(n - x).
\]

On the other hand, from above we know that 
\( m(G - S) = n_p = I - x \), and so we get
\[
f_1(x) = f(1, 1, \cdots, 1, n - x - p + 1, x)
= \left( \frac{I - x}{2} \right) + \left( \frac{x}{2} \right) + x(n - x).
\]

It is easy to see that \(|S| = x \geq 1\) and \( n_p = I - x \geq 1 \), i.e., \( 1 \leq x \leq I - 1 \).

In order to get the maximum value of \( f_1(x) \), we solve the following nonlinear integer programming.
\[
\max f_1(x) = \left( \frac{I - x}{2} \right) + \left( \frac{x}{2} \right) + x(n - x)
\]
\[
s.t \left\{ \begin{aligned}
1 \leq x & \leq I - 1 \\
x & \in Z,
\end{aligned} \right.
\]
where \( Z \) is the set of positive integers.

Since \( f_1'(x) = n - I \), by Lemma 2.1 we know that 
\( f_1'(x) \geq 0 \), and so \( f_1(x) \) is an increasing function in the interval \( 1 \leq x \leq I - 1 \). Then we have
\[
\max f_1(x) = f(I - 1) = \frac{1}{2} I(I - 1) + (I - 1)(n - I),
\]
i.e.,
\[
\max |E(G)| = \frac{1}{2} I(I - 1) + (I - 1)(n - I).
\]

This completes the proof.

It is easy to see that, when \( I(G) = n \), the maximum network \( G \) is just the complete graph \( K_n \).

3 Construction and examples

From Section 2 we know the size of the maximum network with prescribed order and integrity. In the following, we introduce a method for constructing such maximum network \( G = (V, E) \) with order \( n \) and integrity \( I(G) = I \).

Construction:

1. If \( I = n \), construct the complete graph \( K_n \).
2. If \( 2 \leq I \leq n - 1 \), then the construction is as follows:

   Step 1 Construct the complete graph \( K_I \).

   Step 2 Construct the null graph \( E_{n-I} \), such that
   \[
   V(E_{n-I}) \cap V(K_I) = \emptyset,
   \]

   Step 3 Arbitrarily select any \( I - 1 \) vertices in graph \( K_I \) and join these vertices to all the vertices in \( E_{n-I} \).

Thus we get the graphs satisfying the requirements. Clearly, when \( 2 \leq I \leq n - 1 \), the maximum network obtained by the above method is not unique. In the following we give two examples.

Example 3.1 Consider a graph \( G \) with \( n = |V(G)| = 6 \), \( I(G) = 6 \). Then we use (1) of the Construction to construct a complete graph with \( \max |E(G)| = 15 \), as shown in (a) of Figure 1.

Example 3.2 Consider a graph \( G \) with \( n = |V(G)| = 6 \), \( I(G) = 4 \). Then we use (2) of the Construction to construct a graph with \( \max |E(G)| = 12 \), as shown in (b) of Figure 1.

Figure 1. Maximum networks with given order and integrity

4 The trees of minimum integrity with given order and maximum degree

In this section, we determine the minimum integrity of trees with given order and maximum degree. Meanwhile, a method for constructing such trees is presented. The following lemmas are used later.

Lemma 4.1 Let \( G = (V, E) \) be a graph with order \( n \) and integrity \( I(G) = I \). Then for any positive integers \( m, n \) such that \( m \geq n \) and \( s \), there exists an integer \( r \) \((0 \leq r \leq n - 1)\) such that \( m = sn + r \).
Lemma 4.2 If $H$ is a connected spanning subgraph of a connected graph $G$, then $I(G) \geq I(H)$.

Lemma 4.3 If $T$ is a tree with maximum degree $\Delta$ and order $n$, then $T$ has at most $\lfloor \frac{n-2}{\Delta-1} \rfloor$ vertices of degree $\Delta$.

Proof. Let $V' = \{ v : d_T(v) = \Delta \}$ and $|V'| = x$. In order to have the number of vertices of degree $\Delta$ achieve the maximum, without loss of generality, we assume that $T[V']$ is a path, and so we have $(n - x) \geq \Delta x - 2(x - 1)$, i.e., $x(\Delta - 1) \leq n - 2$. Noticing that $x$ is an positive integer, we have that $x \leq \lfloor \frac{n-2}{\Delta-1} \rfloor$.

Definition 4.1 A tree is called a saturated tree with $p$ vertices of maximum degree $\Delta$ if the following conditions are satisfied:

1. $T$ has $p$ vertices of degree $\Delta$,
2. besides all the above vertices, the other vertices in $T$ are all leaves.

Example 4.1 Consider a tree $T$ with order $n = 14$ and $\Delta = 5$. So we have $\lfloor \frac{n-2}{\Delta-1} \rfloor = \lfloor \frac{14-2}{5-1} \rfloor = 3$, and thus the saturated tree is constructed as follows:

Step 1 Construct a path $P_3$ of length 2 such that its vertices are labelled as $v_1, v_2, v_3$ from left to right.

Step 2 Join $v_1$ to four new vertices $v_4, v_5, v_6, v_7$, and join $v_3$ to another four new vertices $v_8, v_9, v_{10}, v_{11}$, then join $v_2$ to three new vertices $v_{12}, v_{13}, v_{14}$. The tree is shown in Figure 2.

Figure 2. A saturated tree with three vertices of degree $\Delta = 5$

Theorem 4.1 Let $T$ be a tree with order $n$ ($n \geq 3$) and maximum degree $\Delta$ ($\Delta \geq 2$). Then the minimum integrity of $T$ is

$$
\min_{T \in \mathcal{T}[n, \Delta]} I(T) = \begin{cases} 
\lfloor \frac{n-2}{\Delta-1} \rfloor + 1, & \text{if } d(\frac{n-2}{\Delta-1}) < \lfloor \frac{n-2}{\Delta-1} \rfloor \\
\lfloor \frac{n-2}{\Delta-1} \rfloor + 2, & \text{if } d(\frac{n-2}{\Delta-1}) \geq \lfloor \frac{n-2}{\Delta-1} \rfloor
\end{cases}
$$

where $\mathcal{T}[n, \Delta] = \{ T : |V(T)| = n, \Delta(T) = \Delta \}$, $\lfloor n-2 \rfloor$ denotes the remainder of $n-2$ divided by $\Delta - 1$.

Proof. We distinguish two cases:

Case 1. If $d(\frac{n-2}{\Delta-1}) < \lfloor \frac{n-2}{\Delta-1} \rfloor$, then we construct a tree as follows:

1. Construct a saturated tree $T'$ with $n - d(\frac{n-2}{\Delta-1})$ vertices and with $\lfloor \frac{n-2}{\Delta-1} \rfloor$ vertices of degree $\Delta$.

2. It is easy to see that $T'$ has $(\lfloor \frac{n-2}{\Delta-1} \rfloor - 1)$ $\Delta$-edges. Arbitrarily select $d(\frac{n-2}{\Delta-1})$ $\Delta$-edges and subdivide them, and we thus get a tree $T$ with order $n$ and $\lfloor \frac{n-2}{\Delta-1} \rfloor$ vertices of degree $\Delta$, such that $d(\frac{n-2}{\Delta-1}) < \lfloor \frac{n-2}{\Delta-1} \rfloor$. From the construction we know that the vertex subset $S = \{ v : d_T(v) = \Delta \}$ is an $I$-set of $T$ such that $m(T - S) = 1$ and $|S| = \lfloor \frac{n-2}{\Delta-1} \rfloor$. Hence

$$
I(T) = |n - 2\frac{\Delta-1}{\Delta-1}| + 1.
$$

The minimality is obvious.

Case 2. If $d(\frac{n-2}{\Delta-1}) \geq \lfloor \frac{n-2}{\Delta-1} \rfloor$, we distinguish two subcases:

Subcase 2.1 If $d(\frac{n-2}{\Delta-1}) = \lfloor \frac{n-2}{\Delta-1} \rfloor$, we construct a tree as follows:

1. Construct a saturated tree $T_1$ with $n - d(\frac{n-2}{\Delta-1})$ vertices and with $\lfloor \frac{n-2}{\Delta-1} \rfloor$ vertices of degree $\Delta$.

2. It is easy to see that there exist $(\lfloor \frac{n-2}{\Delta-1} \rfloor - 1)$ $\Delta$-edges in $T_1$. First, subdivide them, then randomize select one leaf-edge and also subdivide it. We thus get a new tree $T$ with order $n$ and $\lfloor \frac{n-2}{\Delta-1} \rfloor$ vertices of degree $\Delta$. So, by the construction of the tree $T$ we know that the vertex subset $S = \{ v : d_T(v) = \Delta \}$ is an $I$-set of $T$ such that $|S| = \lfloor \frac{n-2}{\Delta-1} \rfloor$ and $m(T - S) = 2$. Thus we have

$$
I(T) = |n - 2\frac{\Delta-1}{\Delta-1}| + 2.
$$

Subcase 2.2 If $d(\frac{n-2}{\Delta-1}) > \lfloor \frac{n-2}{\Delta-1} \rfloor$, we construct a tree as follows:

1. Construct a saturated tree $T_1$ with order $n - d(\frac{n-2}{\Delta-1})$ and with $\lfloor \frac{n-2}{\Delta-1} \rfloor$ vertices of degree $\Delta$. It is obvious that $T$ has $(\Delta - 2)\lfloor \frac{n-2}{\Delta-1} \rfloor + 2$ leaves.

2. Subdivide every $\Delta$-edge of $T_1$, and we get a new tree $T_2$ with

$$
n - d(\frac{n-2}{\Delta-1}) + \lfloor \frac{n-2}{\Delta-1} \rfloor - 1
$$

vertices and with $(\Delta - 2)\lfloor \frac{n-2}{\Delta-1} \rfloor + 2$ leaves. By Lemma 4.1 we know that

$$
d(\frac{n-2}{\Delta-1}) \leq \Delta - 2.
$$
So, we have
\[
d(\frac{n-2}{\Delta-1}) - \lfloor \frac{n-2}{\Delta-1} \rfloor + 1 < (\Delta-2)\lfloor \frac{n-2}{\Delta-1} \rfloor + 2.
\]

Then we arbitrarily select \(d(\frac{n-2}{\Delta-1}) - \lfloor \frac{n-2}{\Delta-1} \rfloor + 1\) leaf-edges. By subdividing them, we get a new tree \(T\) with order \(n\) and with \(\lfloor \frac{n-2}{\Delta-1} \rfloor\) vertices of degree \(\Delta\). From the construction we know that the vertex subset \(S = \{v : d_T(v) = \Delta\}\) is an \(I\)-set of \(T\) such that \(|S| = \lfloor \frac{n-2}{\Delta-1} \rfloor\) and \(m(T - S) = 2\). So we have
\[
I(T) = \lfloor \frac{n-2}{\Delta-1} \rfloor + 2.
\]

It is easy to see that this theorem gives a method to construct trees with minimum integrity when its order and maximum degree are given.

**Remark** From the proof of theorem 4.1, it is easy to see that the tree of the minimum integrity with given order and maximum degree is not unique.

**Example 4.2** Denote by \(T[n, I]\) the set of trees of order \(n\) and integrity \(I\).

1. Construct a tree \(T\) with order 15 and \(\Delta = 4\) such that \(T\) has minimum integrity in \(T[15, 4]\). Since \(n = 15\), \(\Delta = 4\) and \(d(\frac{n-2}{\Delta-1}) = d(\frac{13}{3}) = 1 < 4 = \lfloor \frac{13}{3} \rfloor = \lfloor \frac{n-2}{\Delta-1} \rfloor\), we know that the tree \(T\) is constructed as shown in Figure 3.

2. Construct a tree \(T\) with order 21 and \(\Delta = 6\) such that \(T\) has minimum integrity in \(T[21, 6]\). Since \(n = 21\), \(\Delta = 6\) and \(d(\frac{n-2}{\Delta-1}) = d(\frac{19}{5}) = 4 > 3 = \lfloor \frac{19}{5} \rfloor = \lfloor \frac{n-2}{\Delta-1} \rfloor\), we know that the tree \(T\) is constructed as shown in Figure 4.

3. Construct a tree \(T\) with order 23 and \(\Delta = 7\) such that \(T\) has minimum integrity in \(T[23, 7]\). Since \(n = 23\), \(\Delta = 7\) and \(d(\frac{n-2}{\Delta-1}) = d(\frac{21}{4}) = 3 = \lfloor \frac{21}{4} \rfloor = \lfloor \frac{n-2}{\Delta-1} \rfloor\), we know that the tree \(T\) is constructed as shown in Figure 5.

**Figure 3. A tree of minimum integrity with order 15 and \(\Delta = 4\)**

**Figure 4. A tree of minimum integrity with order 21 and \(\Delta = 6\)**

**Figure 5. A tree of minimum integrity with order 23 and \(\Delta = 7\)**

The above theorem gives the minimum integrity of a tree \(T\) with given order and \(\Delta\). It is well-known that any connected graph has a spanning tree, and so we have the following corollary.

**Corollary 4.1** If \(G = (V, E)\) is a connected graph such that \(|V(G)| = n \geq 3\) and \(\Delta(G) = \Delta \geq 2\), then we have

\[
\min_{G \in G[n, \Delta]} I(G) = \begin{cases} 
\frac{n-2}{\Delta-1} + 1, & \text{if } d(\frac{n-2}{\Delta-1}) < \lfloor \frac{n-2}{\Delta-1} \rfloor \\
\lfloor \frac{n-2}{\Delta-1} \rfloor + 1, & \text{if } d(\frac{n-2}{\Delta-1}) \geq \lfloor \frac{n-2}{\Delta-1} \rfloor
\end{cases}
\]

where \(G[n, \Delta] = \{G : |V(G)| = n, \Delta(G) = \Delta\}\), \(d(\frac{n-2}{\Delta-1})\) denotes the remainder of \(n - 2\) divided by \(\Delta - 1\).

**Proof.** From Lemma 4.2, we know that \(I(G) \geq I(T)\), where \(T\) is the spanning tree of \(G\). It follows from Theorem 4.1 that

\[
\min_{G \in G[n, \Delta]} I(G) = \begin{cases} 
\frac{n-2}{\Delta-1} + 1, & \text{if } d(\frac{n-2}{\Delta-1}) < \lfloor \frac{n-2}{\Delta-1} \rfloor \\
\lfloor \frac{n-2}{\Delta-1} \rfloor + 1, & \text{if } d(\frac{n-2}{\Delta-1}) \geq \lfloor \frac{n-2}{\Delta-1} \rfloor
\end{cases}
\]

where \(G[n, \Delta] = \{G : |V(G)| = n, \Delta(G) = \Delta\}\), \(d(\frac{n-2}{\Delta-1})\) denotes the number of remainder of \(n - 2\) divided by \(\Delta - 1\).

**5 Conclusion**

The robustness of a distributed system of computers can be represented by the integrity of the graph describing the network. The authors present and prove a formula to calculate the maximum number of edges in a network of given integrity. Two construction methods for such networks are given, respectively to construct the maximum network with a given integrity and the minimum integrity network when the network graph is a tree.
parameter, integrity has been studied extensively, but there are many problems remaining unsolved. One interesting problem is: If $G = (V, E)$ is a connected graph with order $n$ and integrity $I([\sqrt{2n + 1}] - 1 \leq I \leq n - 1)$, then $\min_{G \in G(n, I)} |E(G)| = ?$ Another interesting problem is how to construct such networks.

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