Positively Curved Cubic Plane Graphs Are Finite

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Abstract: Let \( G \) be an infinite plane graph such that \( G \) is locally finite and every face of \( G \) is bounded by a cycle. Then \( G \) is said to be positively curved if, for every vertex \( x \) of \( G \),

\[ 1 - \frac{d(x)}{2} + \sum_{x \in F} \frac{1}{|F|} > 0, \]

where the summation is taken over all facial cycles \( F \) of \( G \) containing \( x \) and \( |F| \) denotes the number of vertices in \( F \). Note that if \( G \) is positively curved then the maximum degree of \( G \) is at most 5. As a discrete analog of a result in Riemannian geometry, Higuchi conjectured that if \( G \) is positively curved then \( G \) is finite. In this

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1 INTRODUCTION

The graphs considered in this paper are simple, but may be finite or infinite. Let $G$ be a graph. Then $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$, respectively. We use $|G|$ to denote the number of vertices of $G$. For any $x \in V(G)$, $d_G(x)$ denotes the number of edges of $G$ incident with $x$. (We use $d(x)$ if no confusion arises.) We say that $G$ is cubic if $d(x) = 3$ for all $x \in V(G)$. We say that $G$ is locally finite if $d(x)$ is finite for all $x \in V(G)$. If there is no danger of confusion, we write $x \in G$ instead of $x \in V(G)$. A cycle in $G$ is a finite connected subgraph of $G$ in which every vertex has degree 2. Let $X \subseteq V(G)$. Then $G - X$ denotes the graph obtained from $G$ by deleting $X$ and all edges of $G$ incident with vertices in $X$. If $G$ is connected and $G - X$ is not connected, then $X$ is called a vertex cut of $G$. If $X$ is a vertex cut of $G$ and $|X| = k$, then $X$ is called a $k$-cut. We say that a vertex $x \in V(G) - X$ is adjacent to $X$ if $x$ is adjacent to some vertex in $X$.

For subgraphs $G$ and $H$ of a graph, we use $G \cup H$ and $G \cap H$ to denote the union and intersection of $G$ and $H$, respectively. A plane graph is a graph drawn in the plane with no pair of edges crossing. The vertices and edges incident with a common face of a plane graph are said to be cofacial. Let $G$ be a plane graph. We say that a face of $G$ is bounded by a cycle if the edges of $G$ incident with that face induce a cycle in $G$, and such a cycle is called a facial cycle of $G$. Let $C$ be a cycle in a plane graph. Then we can speak of two orientations on $C$: clockwise orientation and counter-clockwise orientation. Let $u, v$ be distinct vertices of $C$. We use $C[u, v]$ to denote the clockwise path in $C$ from $u$ to $v$. We use $C(u, v)$ to denote the graph obtained from $C[u, v]$ by deleting $u$ and $v$. We define $C[u, v]$ and $C(u, v)$ in the obvious way.

In [2], the curvature of a plane graph is introduced as a discrete analog of the sectional curvature of a Riemannian manifold, and a criterion is given for the hyperbolicity of a plane graph. For more details, see [2] and the references in [2].

Let $G$ be a plane graph (finite or infinite) such that (1) $G$ is locally finite and (2) every face of $G$ is bounded by a cycle. Then the combinatorial curvature of $G$ is the function $\Phi_G$ from $V(G)$ to the set of real numbers such that, for any $x \in V(G)$,

$$\Phi_G(x) = 1 - \frac{d(x)}{2} + \sum_{x \in F} \frac{1}{|F|},$$

where the summation is taken over all facial cycles of $G$ containing $x$. See Figure 1 for an example. We say that a vertex $x$ of $G$ is non-positive if $\Phi_G(x) \leq 0$. If $\Phi_G(x) > 0$ for all $x \in V(G)$, then we say that $G$ is positively curved.
As pointed out in [2], \( \Phi_G(x) \) may be interpreted as the degree of difficulty for tiling the plane at \( x \), and it is dual to another curvature introduced by Gromov [1]. Higuchi in [2] proves that under some minor requirements, if \( \Phi_G(x) < 0 \) for all \( x \in V(G) \) then there exists a constant \( \epsilon > 0 \) such that \( \Phi_G(x) < -\epsilon \) for all \( x \in V(G) \). This is then used to derive a discrete analog of a fact in Riemannian geometry concerning isoperimetric inequalities.

The conjecture below is posed in [2] as a discrete analog of the following result of Myers [3]: A complete Riemannian manifold with Ricci curvature bounded below by a positive number is compact and has finite fundamental group.

**Conjecture 1.1.** Let \( G \) be a locally finite plane graph such that every face of \( G \) is bounded by a cycle. If \( G \) is positively curved, then \( G \) is a finite graph.

The plane graph in Figure 2 is obtained from two vertex disjoint cycles \( u_1 u_2 \cdots u_n u_1 \) and \( v_1 v_2 \cdots v_n v_1 \) by adding a perfect matching \( \{u_i v_i : i = 1, 2, \ldots, n\} \). It is easy to verify that this graph is positively curved. Hence, there exist arbitrarily large cubic graphs which are positively curved. This example suggests that Conjecture 1.1 is not easy to prove.

Higuchi verified Conjecture 1.1 for some special classes of graphs, and he noted that his method brings no insight to Conjecture 1.1. Note that if \( G \) is positively curved then \( d(x) \leq 5 \) for all \( x \in V(G) \). The main result of this paper is the following, which establishes Conjecture 1.1 for all cubic graphs. We believe that our method offers a possible approach to establish Conjecture 1.1 completely.

**Theorem 1.1.** Let \( G \) be a cubic plane graph such that every face of \( G \) is bounded by a cycle. If \( G \) is positively curved then \( G \) is a finite graph.

**FIGURE 1.** \( \Phi_G(x) = 1 - 3/2 + (1/3 + 1/4 + 1/5) = 17/60. \)

**FIGURE 2.** A positively curved graph on \( 2n \) vertices, \( n \geq 3.\)
The main idea of our proof is as follows. Assume (by way of contradiction) that $G$ is infinite. First, we prove the existence of an infinite sequence $(C_0, C_1, \ldots)$ of disjoint cycles in $G$ which captures certain structural information of $G$. This is done without requiring $G$ be cubic. We then assume that $G$ is positively curved, and derive a contradiction by showing that, for all sufficiently large $n$, $|C_n| > |C_{n+1}|$. This is done through case analysis.

This paper is organized as follows. In Section 2, we first show how to produce an infinite sequence of cycles mentioned above. We then use that sequence to derive a plane embedding of the same graph (with the same curvature function) that is easier to deal with. In Section 3, we introduce necessary notation and derive further structural information about positively curved cubic plane graphs. In Sections 4–7, we prove Theorem 1.1.

2. NICE SEQUENCES

The main objective of this section is to derive some useful structural information about infinite plane graphs. To this end, we need the following convenient concept.

**Definition 2.1.** Let $H$ be a subgraph (finite or infinite) of a graph $G$ (finite or infinite). An $H$-bridge of $G$ is a subgraph of $G$ which is induced by either (1) an edge $e \in E(K) - E(H)$ with both incident vertices on $H$ or (2) edges in a component $D$ of $G - V(H)$ and edges from $D$ to $H$. If $B$ is an $H$-bridge of $G$, then the vertices in $V(H) \cap V(B)$ are attachments of $B$ on $H$. Note that an $H$-bridge of $G$ may be infinite. For any $S \subseteq V(G)$, we may view $S$ as a graph with vertex set $S$ and no edges, and hence, we may speak of $S$-bridges.

In Figure 3, $H = uwy$ is a path. The $H$-bridges of $G$ are the subgraphs induced by the following sets of edges: $\{uv, vw\}$, $\{wx, xy\}$, $\{zu, zw, zy\}$, and $\{uy\}$.

We now turn our attention to the description of a “nice” sequence of cycles. Let $G$ be an infinite plane graph such that $G$ is locally finite and every face of $G$ is bounded by a cycle. Let $F$ be a facial cycle of $G$ and let $R(F)$ denote the closure of the face of $G$ bounded by $F$. (Thus, $F$ is the boundary of $R(F)$.) For any cycle $C$ in $G$, we define $R_F(C)$ as follows. By the Jordan curve theorem, $C$ divides the plane into two closed regions whose intersection is $C$, and we use $R_F(C)$ to denote the closed region containing $R(F)$. Hence, $R_F(F) = R(F)$. For any cycle $C$ in $G$, we use $G_F(C)$ to denote the subgraph of $G$ contained in $R_F(C)$.

![Figure 3. A path H and its bridges.](image-url)
Definition 2.2. Let $G$ be an infinite plane graph such that $G$ is locally finite and every face of $G$ is bounded by a cycle, and let $F$ be a facial cycle of $G$. A sequence of disjoint cycles $(C_0, C_1, \ldots)$ in $G$ is called a nice sequence starting with $F$ if the following conditions hold:

1. $C_0 = F$,
2. for $i \geq 0$, $R_F(C_i) \subseteq R_F(C_{i+1})$ (and hence, $G_F(C_i) \subseteq G_F(C_{i+1})$),
3. for $i \geq 0$, every $(G_F(C_i) \cup C_{i+1})$-bridge of $G_F(C_{i+1})$ has at most one attachment on $C_{i+1}$, and
4. for $i \geq 0$, $G - V(G_F(C_i))$ is infinite.

Figure 4 illustrates a nice sequence $(C_0, C_1, \ldots)$, where, for $i \geq 0$, $R_F(C_i)$ is the closed disc bounded by $C_i$. The shaded regions represent subgraphs which may be finite or infinite. Notice that a $(G_F(C_i) \cup C_{i+1})$-bridge $B$ of $G_F(C_{i+1})$ may be infinite, but the vertices and edges of $B$ cofacial with a vertex of $C_{i+1}$ form a finite subgraph $B^*$ of $G$. In fact, $B^*$ is the union of two finite paths, because every face of $G$ is bounded by a cycle.

We are now ready to state and prove the main result of this section.

Theorem 2.1. Let $G$ be an infinite plane graph such that $G$ is locally finite and every face of $G$ is bounded by a cycle, and let $F$ be a facial cycle of $G$. Then $G$ has a nice sequence starting with $F$.

Proof. Let $C_0 = F$, let $G_F(C_0) = C_0$, and let $R_F(C_0)$ be the closure of the face bounded by $F$. Suppose that we have constructed $(C_0, \ldots, C_k)$ for some $k \geq 0$ such that

1. $C_0 = F$,
2. for $0 \leq i \leq k - 1$, $R_F(C_i) \subseteq R_F(C_{i+1})$ and $G_F(C_i) \subseteq G_F(C_{i+1})$,
3. for $0 \leq i \leq k - 1$, every $(G_F(C_i) \cup C_{i+1})$-bridge of $G_F(C_{i+1})$ has at most one attachment on $C_{i+1}$, and
4. for $0 \leq i \leq k$, $G - V(G_F(C_i))$ is infinite.
The remainder of this proof shows how to construct the next cycle for the desired sequence.

Consider the graph \( H := G - V(G_F(C_k)) \). Note that \( H \) needs not be connected, but every block of \( H \) contains a vertex that is cofacial with some vertex of \( C_k \).

Since \( G \) is locally finite and \(|C_k|\) is finite, there are only finitely many facial cycles of \( G \) intersecting \( C_k \). Since all faces of \( G \) are bounded by cycles, \( H \) has only finitely many blocks.

Therefore, since \( H \) is infinite, some block of \( H \), say \( B \), is infinite. Let \( C_k \) denote the subgraph of \( H \) consisting of vertices and edges of \( B \) cofacial with a vertex of \( C_k \). Observe that \( C_k \) is finite; because \( C_k \) is finite, \( G \) is locally finite, and every face of \( G \) is bounded by a cycle. Since \( B \) is 2-connected, \( C_k \) is a cycle.

Obviously, \( R_F(C_k) \subseteq R_F(C_{k+1}) \) and \( G_F(C_k) \subseteq G_F(C_{k+1}) \). Because \( B \) is a block of \( G - V(G_F(C_k)) \), every \((G_F(C_k) \cup C_{k+1})\)-bridge of \( G_F(C_{k+1}) \) has at most one attachment on \( C_{k+1} \). Because \( B \) is infinite and \( C_{k+1} \) is finite, \( B - V(C_{k+1}) \) is infinite. Hence \( G - V(G_F(C_{k+1})) \) is infinite. So the sequence \((C_0, \ldots, C_{k+1})\) satisfies (1)–(4) above with \( k + 1 \) replacing \( k \). This process can be continued with \((C_0, \ldots, C_{k+1})\) replacing \((C_0, \ldots, C_k)\). Hence, the desired nice sequence \((C_0, C_1, \ldots)\) exists.

To facilitate later discussions, we will work with a “nice” embedding of a plane graph \( G \) which has the same combinatorial curvature as \( G \). Such an embedding is guaranteed to exist by a nice sequence.

**Theorem 2.2.** Let \( G \) be an infinite plane graph such that \( G \) is locally finite and every face of \( G \) is bounded by a cycle, and let \( F \) be a facial cycle of \( G \), and let \((C_0, C_1, C_2, \ldots)\) be a nice sequence in \( G \) starting with \( F \). Then \( G \) has an embedding \( G' \) in the plane such that

1. \( F \) is a facial cycle of \( G' \),
2. for \( i \geq 0 \), \( G_F(C_i) \) is contained in the closed disc bounded by \( C_i \), and
3. \( G \) and \( G' \) have the same combinatorial curvature.

**Proof.** Consider the graphs \( H_i := G_F(C_{i+1}) - (V(G_F(C_i)) - V(C_i)) \) for all \( i \geq 0 \). Each \( H_i \) is a subgraph of \( G \). Hence, \( H_i \) is a plane graph and both \( C_i \) and \( C_{i+1} \) are facial cycles of \( H_i \). Therefore, \( H_i \) has an embedding \( H'_i \) in the plane such that

(a) for any cycle \( C \) in \( H_i \), \( C \) is a facial cycle of \( H'_i \) if, and only if, \( C \) is a facial cycle of \( H_i \),
(b) the face of \( H'_i \) bounded by \( C_i \) is an open disc in the plane,
(c) the face of \( H'_i \) bounded by \( C_{i+1} \) is an unbounded region in the plane.

By assembling the embeddings \( H'_i \), for all \( i \geq 0 \), we obtain an embedding \( G' \) of \( G \) satisfying (1), (2), and (3).
We say that \( G \) is *nicely embedded* with respect to a nice sequence \((C_0, C_1, \ldots)\) if, for each \( i \geq 0 \), \( G_F(C_i) \) is contained in the closed disc bounded by \( C_i \).

3. NOTATION AND CONVENTION

Let \( G \) be an infinite plane graph such that \( G \) is cubic and every face of \( G \) is bounded by a cycle. Let \( v \) be a vertex of \( G \). Let \( F_1, F_2, \) and \( F_3 \) be the facial cycles of \( G \) containing \( v \), and assume that \(|F_1| \leq |F_2| \leq |F_3|\). We define \( \ell(v) = (|F_1|, |F_2|, |F_3|) \). If \(|F_1| \geq m_1, |F_2| \geq m_2, \) and \(|F_3| \geq m_3\), then we write \( \ell(v) \geq (m_1, m_2, m_3) \). For a vertex \( v \) of \( G \) with \( \ell(v) = (m_1, m_2, m_3) \), \( \Phi_G(v) > 0 \) if, and only if, \( 1/m_1 + 1/m_2 + 1/m_3 > 1/2 \). The following lemma is easy to verify.

**Lemma 3.1.** Let \( G \) be a cubic, infinite, plane graph such that every face of \( G \) is bounded by a cycle. Let \( T \) be the set consisting of the following triples: \((3, 7, 42), (3, 8, 24), (3, 9, 18), (3, 10, 15), (3, 11, 14), (3, 12, 12), (4, 5, 20), (4, 6, 12), (4, 7, 10), (4, 8, 8), (5, 5, 10), (5, 6, 8), (5, 7, 7), \) or \((6, 6, 6)\). If \( v \in V(G) \) such that \( \ell(v) \geq (m_1, m_2, m_3) \) for some \((m_1, m_2, m_3) \in T \), then \( v \) is non-positive.

We note that \((3, 7, 42), (3, 8, 24), (3, 9, 18), \) and \((4, 5, 20)\) are not needed for our arguments; their inclusion is for the sake of completeness. Throughout the rest of the paper, we will use Lemma 3.1 to derive contradictions by showing that in a positively curved cubic infinite plane graph, there is a non-positive vertex \( v \). When understood, Lemma 3.1 will not be referred explicitly.

Let \( G \) be a cubic, infinite, plane graph such that every face of \( G \) is bounded by a cycle. Let \( F \) be a facial cycle of \( G \). By Theorem 2.1, \( G \) has a nice sequence \((C_0, C_1, \ldots)\) starting with \( F \). By Theorem 2.2, we may assume that \( G \) is nicely embedded with respect to \((C_0, C_2, \ldots)\). A vertex \( v \) of \( G \) is called an *in-vertex* (respectively, *out-vertex*) if \( v \in C_i \) for some \( i \geq 1 \) (respectively, \( i \geq 0 \)) and \( v \) is incident with an edge contained in the annulus region between \( C_i \) and \( C_{i+1} \) (respectively, \( C_{i-1} \)). If \( v \) is an in-vertex on \( C_i \), then we use \( A(v), L(v), R(v) \) to denote the facial cycles of \( G \) containing \( v \), where \( A(v) \) is between \( C_i \) and \( C_{i+1} \) and \( A(v), R(v), L(v) \) occur in that clockwise order around \( v \). (Intuitively, \( A(v) \) is above \( v \), \( L(v) \) is to the left of \( v \), and \( R(v) \) is to the right of \( v \).) If \( w \) is an out-vertex on \( C_i \), then we use \( B(w), L(w), R(w) \) to denote the facial cycles of \( G \) containing \( w \), where \( B(w) \) is between \( C_i \) and \( C_{i-1} \) and \( B(w), L(w), R(w) \) occur around \( w \) in that clockwise order. (Again, \( B(w) \) is below \( w \), \( L(w) \) is to the left of \( w \), and \( R(w) \) is to the right of \( w \).) See Figure 5.

![Figure 5](https://example.com/figure5.png)

**FIGURE 5.** \( v \) is an in-vertex and \( w \) is an out-vertex.
By the choice of \((C_0, C_1, \ldots)\), we have the following two observations which will be used frequently (often without explicit reference).

**Lemma 3.2.** Let \(G\) be a cubic, infinite, plane graph such that every face of \(G\) is bounded by a cycle. Let \((C_0, C_1, \ldots)\) be a nice sequence in \(G\), and assume that \(G\) is nicely embedded with respect to \((C_0, C_1, \ldots)\). Then for any in-vertex \(v\), \(|L(v)| \geq 4 \leq |R(v)|\); and for any out-vertex \(w\), \(|B(w)| \geq 5\).

**Lemma 3.3.** Let \(G\) be a cubic, infinite, plane graph such that every face of \(G\) is bounded by a cycle. Let \((C_0, C_1, \ldots)\) be a nice sequence in \(G\), and assume that \(G\) is nicely embedded with respect to \((C_0, C_1, \ldots)\). Then for any facial cycle \(F\) of length 4 and for any \(i \geq 1\), \(|F \cap C_i| \neq 1\).

The next lemma allows us to discard certain 2-cuts of \(G\) in the proof of Theorem 1.1.

**Lemma 3.4.** If there is a positively curved, cubic, infinite, plane graph, then there is a positively curved, cubic, infinite, plane graph \(G\) such that

1. \(G\) has a nice sequence \((C_0, C_1, \ldots)\), and
2. for any \(k \geq 1\) and for any 2-cut \(T\) of \(G\) contained in \(V(C_k)\), \(\bigcup_{0 \leq i \leq k-1} C_i\) and \(\bigcup_{i \geq k+1} C_i\) belong to different components of \(G - T\).

**Proof.** Let \(G\) be a positively curved, cubic, infinite, plane graph such that every face of \(G\) is bounded by a cycle. Then by Definition 2.2, \(G\) has a nice sequence \((C_0, C_1, C_2, \ldots)\). Suppose that there is some \(k \geq 1\) and a 2-cut \(T = \{u, v\}\) of \(G\) contained in \(V(C_k)\) such that \(\bigcup_{0 \leq i \leq k-1} C_i\) and \(\bigcup_{i \geq k+1} C_i\) belong to the same component of \(G - T\). Let \(B\) denote the \(T\)-bridge of \(G\) not containing \(C_k\). We may assume that \(T\) is chosen so that \(B\) is maximal. Then, since \(G\) is cubic, \(B\) contains exactly one neighbor of \(u\) and exactly one neighbor of \(v\). So let \(H\) denote the plane graph obtained from \(G\) by replacing \(B\) with the edge \(uv\). Then \(H\) is a cubic, infinite, plane graph. Moreover, for each vertex \(x\) of \(H\), the length of any facial cycle of \(H\) containing \(x\) is not longer than the corresponding facial cycle of \(G\). So \(H\) is also positively curved. Since \(|C_k|\) is finite, there are only finitely many 2-cuts contained in \(V(C_k)\). Thus, we can repeatedly perform the above operation to eliminate all 2-cuts contained in \(V(C_k)\) that do not satisfy (2). We can deal with \(C_0, C_1, C_2, \ldots\) in that order, and we see that Lemma 3.4 holds.

The proof of Theorem 1.1 is divided into the following stages. Assume that \(G\) is a positively curved, cubic, infinite, plane graph such that every face of \(G\) bounded by a cycle. Then, by the results in Section 2, \(G\) has a nice sequence \((C_0, C_1, \ldots)\) and we can assume that \(G\) is nicely embedded with respect to that sequence. First, we will show that, for all sufficiently large \(i\), there are at most three vertices of \(C_i\) between any two consecutive in-vertices on \(C_i\). This is done in Section 4. We will then use the result in Section 4 to show that, for all sufficiently
large $i$, there are at most two vertices of $C_i$ between any two consecutive in-vertices on $C_i$, and this is done in Section 5. In Section 6, we will further show that, for all sufficiently large $i$, there are at most one vertex of $C_i$ between any two consecutive in-vertices on $C_i$. Finally, we will complete the proof in Section 7 by showing that, for all sufficiently large $i$, $|C_{i+1}| < |C_i|$.  

4. FOUR VERTICES BETWEEN CONSECUTIVE IN-VERTICES  

For convenience, we assume, throughout this section, that $G$ is a positively curved, cubic, infinite, plane graph. So $G$ contains no non-positive vertices. By Definition 2.2 and Theorem 2.2, $G$ has a nice sequence $(C_0, C_1, \ldots)$ and we may assume that $G$ is nicely embedded with respect to $(C_0, C_1, \ldots)$. 

The main result of this section is the following: if $i$ is large enough, then there are at most three vertices of $C_i$ between any two consecutive in-vertices on $C_i$. This is done through a series of lemmas. For the statement and proof of the first lemma, we refer to Figure 6.

**Lemma 4.1.** Let $i \geq 3$, and let $a_1$ and $a_2$ be consecutive in-vertices on $C_i$ such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$. Let $b_1, b_2$ be the out-vertices on $C_{i-1}$ such that $R(b_1) = L(b_2) = R(a_1)$. Then $C_{i-1}(b_1, b_2) \neq \emptyset$ or $|L(b_1)| = |R(b_2)| = 3$.

**Proof.** Suppose $C_{i-1}(b_1, b_2) = \emptyset$. Since $G$ is cubic, $B(b_1) = B(b_2)$. So let $c_1, c_2$ be the out-vertices on $C_{i-2}$ such that $R(c_1) = B(b_1) = L(c_2)$. By (ii), $|R(b_1)| = |L(b_2)| \geq 8$. Since $C_{i-1}(b_1, b_2) = \emptyset$, $|B(b_1)| = |B(b_2)| \geq 6$.

Therefore, $|L(b_1)| \leq 4$ and $|R(b_2)| \leq 4$; for otherwise, there would exist $i \in \{1, 2\}$ such that $\ell(b_i) \geq (5, 6, 8)$ and $b_i$ is non-positive (by Lemma 3.1). If $|L(b_1)| = |R(b_2)| = 3$, then we have Lemma 4.1. Therefore, we have two cases to consider.

**Case 1.** $|L(b_1)| = |R(b_2)| = 4$.  

![Figure 6. Proof of Lemma 4.1.](image)
Then, by Lemma 3.3, \(|L(b_1) \cap C_{i-1}| \in \{2, 3\} \) and \(|R(b_2) \cap C_{i-1}| \in \{2, 3\} \).

First, assume that \(|L(b_1) \cap C_{i-1}| = 2 = |R(b_2) \cap C_{i-1}| \). Then, since \(G\) is cubic and \(C_{i-1}(b_1, b_2) = \emptyset, |B(b_1)| \geq 8\). Hence \(\ell(b_1) \geq (4, 8, 8)\) and \(b_1\) is non-positive (by (3.1)), a contradiction.

Now assume that \(|L(b_1) \cap C_{i-1}| = 2 \neq 2 \) and \(|R(b_2) \cap C_{i-1}| = 2 \). By symmetry, we may assume the former. See Figure 6(a). Then \(|R(b_1)| = |L(b_2)| \geq 9\), and since \(G\) is cubic and \(C_{i-1}(b_1, b_2) = \emptyset, |B(b_1)| = |B(b_2)| \geq 7\). Thus, \(\ell(b_1) = \ell(b_2) = (4, 7, 9)\); for otherwise, there would exist \(i \in \{1, 2\}\) such that \(\ell(b_i) \geq (4, 8, 9)\) or \((4, 7, 10)\) and \(b_i\) is non-positive. Hence, \(C_{i-2}(c_1, c_2) = \emptyset\) and \(c_2\) is adjacent to \(C_{i-1}\). So \(|B(c_2)| \geq 6\) and \(|R(c_2)| \geq 5\) (because \(|R(b_2) \cap C_{i-1}| = 3\)). If \(|R(c_2)| = 5\) then \(|B(c_2)| \geq 7\), and hence, \(\ell(c_2) \geq (5, 7, 7)\) and \(c_2\) is non-positive, a contradiction. So \(|R(c_2)| \geq 6\).

Then \(\ell(c_2) \geq (6, 6, 7)\) and \(c_2\) is non-positive, a contradiction.

So \(|L(b_1) \cap C_{i-1}| = 3 = |R(b_2) \cap C_{i-1}| \). See Figure 6(b). Hence, \(|R(b_1)| = |L(b_2)| \geq 10\). Since \(|B(b_1)| = |B(b_2)| \geq 6, \ell(b_1) \geq (4, 6, 10), \) and \(\ell(b_2) \geq (4, 6, 10)\). In fact, \(|B(b_1)| = |B(b_2)| = 6\), as otherwise, there would exist \(i \in \{1, 2\}\) such that \(\ell(b_i) \geq (4, 7, 10)\) and \(b_i\) is non-positive. So \(C_{i-2}(c_1, c_2) = \emptyset, |L(c_1)| \geq 5 \leq |R(c_2)|, \) and \(|B(c_1)| = |B(c_2)| \geq 6\). Suppose \(|L(c_1)| = |R(c_2)| = 5\). Then \(|L(c_1) \cap C_{i-2}| = 2 = |R(c_2) \cap C_{i-2}| \). Since \(G\) is 2-connected and \(C_{i-2}\) is cubic and \(c_1, c_2 \in G, |B(c_1)| = |B(c_2)| \geq 8\). Hence \(\ell(c_1) \geq (5, 6, 8)\) and \(c_1\) is non-positive, a contradiction. So \(|L(c_1)| \geq 6 \) or \(|R(c_2)| \geq 6\). Then there exists \(i \in \{1, 2\}\), such that \(\ell(c_i) \geq (6, 6, 6)\) and \(c_i\) is non-positive, a contradiction.

Case 2. \(|L(b_1)| = 4 \) and \(|R(b_2)| = 3\), or \(|L(b_1)| = 3 \) and \(|R(b_2)| = 4\).

By symmetry, we may assume the former. Since \(|R(b_2)| = 3, |L(b_2)| = |R(b_1)| = 9\). First, assume that \(|L(b_1) \cap C_{i-1}| = 3\). See Figure 6(c). Then since \(G\) is cubic and \(C_{i-1}(b_1, b_2) = \emptyset, |B(b_1)| \geq 8\). Therefore, \(\ell(b_1) \geq (4, 8, 9)\) and \(b_1\) is non-positive, a contradiction. So \(|L(b_1) \cap C_{i-1}| = 3\). See Figure 6(d). Then \(|B(b_1)| \geq 7 \) and \(|R(b_1)| \geq 10\). So \(\ell(b_1) \geq (4, 7, 10)\) and \(b_1\) is non-positive, a contradiction.

**Lemma 4.2.** Let \(i \geq 5\), and let \(a_1\) and \(a_2\) be consecutive in-vertices on \(C_i\) such that (i) \(R(a_1) = L(a_2)\) and (ii) \(|C_i(a_1, a_2)| \geq 4\). Let \(b_1\) and \(b_2\) be the out-vertices on \(C_{i-1}\) such that \(R(b_1) = L(b_2) = R(a_1)\). Then

1. \(|C_{i-1}(b_1, b_2)| \neq 1\), and
2. \(|C_{i-1}(b_1, b_2)| \neq 2\) if \(i \geq 7\).

**Proof.** (1) Suppose \(|C_{i-1}(b_1, b_2)| = 1\). Let \(b\) be the only vertex in \(C_{i-1}(b_1, b_2)\). Since \(G\) is 2-connected, \(b\) is an in-vertex and \(B(b_1) = L(b)\) and \(R(b) = B(b_2)\). Let \(c_1, c, c_2\) be the out-vertices on \(C_{i-2}\) such that \(R(c_1) = L(c) = B(b_1)\) and \(L(c_2) = B(b_2)\). See Figure 7.

Note that \(|R(b_1)| = |A(b)| = |L(b_2)| \geq 9, |L(b)| = |B(b_1)| \geq 5, \) and \(|R(b)| = |B(b_2)| \geq 5\). Hence \(\ell(b) \geq (5, 5, 9)\). Therefore, \(|L(b)| = |R(b)| = 5\) and \(|A(b)| = 9\); otherwise, \(\ell(b) \geq (5, 6, 9)\) or \((5, 5, 10)\) and \(b\) is non-positive, a contradiction.
Since $|A(b)| = 9$, $a_i$ is adjacent to $b_i$ for $i = 1, 2$. Since $|L(b)| = |R(b)| = 5$, $|L(b_1)| \geq 5 \leq |R(b_2)|$, $C_{i-2}(c_1, c) = \emptyset = C_{i-2}(c, c_2)$, $b$ is adjacent to $c$, and both $c_1$ and $c_2$ are adjacent to $C_{i-1}$. Therefore, $|B(c_1)| = |B(c)| = |B(c_2)| \geq 7$. So $|L(b_1)| = 5 = |R(b_2)|$, for otherwise, there exists $i \in \{1, 2\}$ such that $\ell(b_i) \geq (5, 6, 9)$ and $b_i$ is non-positive, a contradiction.

Since $G$ is 2-connected, let $c_1', c_2'$ be the in-vertices on $C_{i-2}$ such that $R(c_1') = L(c_2')$, and let $d_1, d_2$ be the out-vertices on $C_{i-3}$ such that $R(d_1) = L(d_2) = B(c_1)$.

We claim that $C_{i-2}(c_1', c_1) = \emptyset$ or $C_{i-2}(c_2, c_2') = \emptyset$. Otherwise, $|C_{i-2}(c_1', c_1)| \geq 5$. Since $i - 2 \geq 3$, it follows from Lemma 4.1 that $|L(d_1)| = 3 = |R(d_2)|$ or $C_{i-1}(d_1, d_2) \neq \emptyset$. So $|B(c)| \geq 10$, and hence $\ell(c) \geq (5, 5, 10)$ and $c$ is non-positive, a contradiction.

By symmetry, we may assume that $C_{i-2}(c_1', c_1) = \emptyset$. Since $|L(b_1)| = 5$ and $a_1$ is adjacent to $b_1$, we have $|A(c_1')| \geq 6$. So $|L(c_1)| = |A(c_1')| = 6$ and $|B(c_1)| = 7$, as otherwise, $c_1$ would be non-positive with $\ell(c_1) \geq (5, 6, 8)$ or $(5, 7, 7)$. Thus $c_1'$ is adjacent to $d_1$, $C_{i-3}(d_1, d_2) = \emptyset$ (and hence $|B(d_1)| \geq 6$), and $|L(d_1)| \geq 5$. Moreover, if $|L(d_1)| = 5$ then $|B(d_1)| \geq 7$. So $\ell(d_1) \geq (5, 7, 7)$ or $(6, 6, 7)$, and hence, $d_1$ is non-positive, a contradiction.

(2) Now suppose $i \geq 7$ and $|C_{i-1}(b_1, b_2)| = 2$. Let $b_3, b_4$ denote the vertices in $C_{i-1}(b_1, b_2)$. By (2) of Lemma 3.4, both $b_3$ and $b_4$ are in-vertices. Without loss of generality, we may assume that $R(b_3) = L(b_4)$. See Figure 8.

Observe that $|R(b_1)| = |L(b_2)| = |A(b_3)| = |A(b_4)| \geq 10$, $|B(b_1)| = |L(b_3)| \geq 5 \leq |R(b_4)| = |B(b_2)|$, and $|R(b_3)| = |L(b_4)| \geq 4$. Therefore, $|R(b_3)| = |L(b_4)| = 4$ and $|L(b_3)| = |B(b_1)| \leq 6 \geq |R(b_4)| = |B(b_2)|$; for otherwise, there would exist...
$i \in \{3, 4\}$ such that $\ell(b_i) \geq (5, 5, 10)$ or $(4, 7, 10)$, and so, $b_i$ is non-positive, a contradiction.

Since $|R(b_3)| = 4$, we let $c_3, c_4 \in V(C_{i-3})$ such that $b_3c_3, b_4c_4 \in E(G)$ and $C_{i-2}(c_3, c_4) = \emptyset$. Since $G$ is cubic and 2-connected, $C_{i-2}(c_4, c_3)$ has at least two in-vertices. We claim that $C_{i-2}(c_4, c_3)$ contains at least two out-vertices. For, suppose $C_{i-2}(c_4, c_3)$ contains at most one out-vertex. Then, since $|L(b_3)| \leq 6 \geq |R(b_4)|$, $C_{i-2}(c_4, c_3)$ contains exactly one out-vertex, and since $G$ is 2-connected and cubic, $|L(b_4)| = 6 = |R(b_4)|$. So $|L(b_4)| \geq 5$. Thus $\ell(b_4) \geq (5, 6, 10)$ and $b_4$ is non-positive, a contradiction. So let $c_1, c_2$ be distinct out-vertices on $C_{i-2}$ such that $R(c_1) = B(b_1)$ and $L(c_2) = B(b_2)$.

We claim that $C_{i-2}(c_1, c_3) = \emptyset = C_{i-2}(c_4, c_2)$. For otherwise, we may assume by symmetry that $C_{i-2}(c_1, c_3) \neq \emptyset$. See Figure 8(a). Because $|L(b_3)| \leq 6$, $C_{i-2}(c_1, c_3)$ consists of only one vertex, say $c_5$. Since $G$ is 2-connected, $c_5$ is an in-vertex. Thus, we see that $|A(c_5)| = 6$, $|L(c_5)| \geq 5$, and $|R(c_5)| \geq 6$. So $|L(c_5)| = 5$ and $|R(c_5)| \leq 7$; otherwise $\ell(c_5) \geq (5, 6, 8)$ or $(6, 6, 6)$, and $c_5$ would be non-positive. Hence $c_5$ is adjacent to a vertex, say $d$, on $C_{i-3}$. Assume for the moment that $C_{i-2}(c_4, c_2) = \emptyset$. Then $|R(c_5)| \geq 7$, and hence $|R(c_5)| = 7$ and $|R(c_4) \cap C_{i-3}| = 2$. Therefore, since $|L(c_5)| = 5$, $|B(d)| \geq 7$. So $\ell(d) \geq (5, 7, 7)$ and $d$ is non-positive, a contradiction. Thus $C_{i-2}(c_4, c_2) \neq \emptyset$. Because $|R(b_4)| \leq 6$, $C_{i-2}(c_4, c_2)$ consists of only one vertex, say $c_6$. Since $G$ is 2-connected, $c_6$ is an in-vertex. Note that $|L(c_6)| = |R(c_3)| \geq 6$ and $|A(c_6)| = 6$. Also note that $|R(c_6)| = 5$, for otherwise $\ell(c_6) \geq (6, 6, 6)$ and $c_6$ would be non-positive. Thus $c_6$ is adjacent to a vertex, say $d'$, on $C_{i-3}$. If $C_{i-3}(d, d') = \emptyset$, then since $|L(c_5)| = |R(c_6)| = 5$, $|B(d)| \geq 8$ and $\ell(d) \geq (5, 6, 8)$, and so, $d'$ would be non-positive. Hence $C_{i-3}(d, d') \neq \emptyset$. Since $|R(c_5)| \leq 7$, $C_{i-3}(d, d')$ consists of only one vertex, say $d''$. Because $|L(c_5)| = 5 = |R(c_6)|$, $|L(d'')| \geq 6 \leq |R(d'')|$. So $\ell(d'') \geq (6, 6, 7)$ and $d''$ is non-positive, a contradiction.

Therefore, let $c_1', c_2'$ denote the in-vertices on $C_{i-2}$ such that $R(c_1') = L(c_2') = B(b_1)$, and let $d_1, d_2$ be the out-vertices on $C_{i-3}$ such that $R(d_1) = L(d_2) = B(b_1)$. See Figure 8(b). Since $i - 2 \geq 5$ and $|C_{i-2}(c_1', c_2')| \geq 4$, it follows from (1) that $|C_{i-3}(d_1, d_2)| \neq 1$. So by Lemma 4.1, $|L(d_1)| = |R(d_2)| = 3$ or $|C_{i-3}(d_1, d_2)| \geq 2$. Thus $|B(c_1)| = |B(c_2)| \geq 10$. Since $|R(c_1)| \geq 5 \leq |L(c_2)|$, $|L(c_1)| \leq 4 \geq |R(c_2)|$, for otherwise, there exists $i \in \{1, 2\}$ such that $\ell(c_i) \geq (5, 5, 10)$ and $c_i$ is non-positive, a contradiction. Also $|L(b_1)| \leq 4 \geq |R(b_2)|$, for otherwise, there exists $i \in \{1, 2\}$ such that $\ell(b_i) \geq (5, 5, 10)$ and $b_i$ is non-positive, a contradiction.

We claim that $|L(b_1)| = 4$ and $|L(b_1) \cap C_{i-1}| = 2$ or $|R(b_2)| = 4$ and $|R(b_2) \cap C_{i-1}| = 2$. Suppose this is false. Then $|R(b_1)| = |L(b_2)| \geq 12$. Hence $|L(b_3)| = |B(b_1)| = 5$; otherwise, $\ell(b_3) \geq (4, 6, 12)$ and $b_3$ would be non-positive. Therefore, $|L(b_1)| = 4$ and $|L(b_1) \cap C_{i-1}| = 3$. Then, since $|L(b_3)| = 5$, $|L(c_1)| \geq 5$, a contradiction.

Without loss of generality, we may assume that $|L(b_1)| = 4$ and $|L(b_1) \cap C_{i-1}| = 2$. See Figure 8(b). Then $|B(b_1)| \geq 6$. In fact, $|B(b_1)| = 6$; otherwise, $\ell(b_1) \geq (4, 7, 10)$ and $b_1$ is non-positive, a contradiction.
If $|L(c_1)| \geq 5$, then $\ell(c_1) \geq (5, 6, 10)$ and $c_1$ is non-positive, a contradiction. So $|L(c_1)| = 4$. Hence $|L(c_1) \cap C_{i-2}| = 2$ and $|B(c_1)| \geq 11$. In fact, $|B(c_1)| = 11$, as otherwise $\ell(c_1) \geq (4, 6, 12)$ and $c_1$ would be non-positive. So $c_1'$ is adjacent to $L(c_1)$ and $C_{i-2}(c_2', c_2') = \emptyset$. Thus $|A(c_1')| \geq 5$, $|A(c_2')| = |R(c_2)| = 4$, and $|A(c_2') \cap C_{i-2}| = 3$.

If $C_{i-3}(d_1, d_2) \neq \emptyset$, then since $i - 2 \geq 5$ and by (1), $|C_{i-3}(d_1, d_2)| \geq 2$. Thus $|B(c_1)| = 11$ implies that $c_2'$ is adjacent to $d_2$. Since $|A(c_2')| = 4$ and $|A(c_2') \cap C_{i-2}| = 3$, $|R(d_2)| = |R(c_2')| \geq 5$. Hence $\ell(d_2) \geq (5, 5, 11)$ and $d_2$ is non-positive, a contradiction. Therefore, $C_{i-3}(d_1, d_2) = \emptyset$. Then by Lemma 4.1, $|L(d_1)| = 3$, and hence $|L(c_1')| \geq 5$. So $\ell(c_1') \geq (5, 5, 11)$ and $c_1'$ is non-positive, a contradiction.

**Lemma 4.3.** Let $i \geq 7$, and let $a_1$ and $a_2$ be consecutive in-vertices on $C_i$ such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$. Let $b_1$ and $b_2$ be the out-vertices on $C_{i-1}$ such that $R(b_1) = L(b_2) = R(a_1)$. Assume that $C_{i-1}(b_1, b_2) = \emptyset$, and let $c_1, c_2$ denote the out-vertices on $C_{i-2}$ such that $R(c_1) = L(c_2) = B(b_1) = B(b_2)$. Then $C_{i-2}(c_1, c_2) \neq \emptyset$.

**Proof.** Suppose for a contradiction that $C_{i-2}(c_1, c_2) = \emptyset$. Let $b_1', b_2'$ denote the in-vertices on $C_{i-1}$ such that $R(b_1') = L(b_2') = B(b_1) = B(b_2)$; let $c_1', c_2'$ be the in-vertices on $C_{i-2}$ such that $R(c_1') = L(c_2') = B(c_1) = B(c_2)$; and let $d_1, d_2$ denote the out-vertices on $C_{i-3}$ such that $R(d_1) = L(d_2) = B(c_1)$. See Figure 9. Since $G$ is 2-connected and cubic, the above vertices are well defined.

Since $C_{i-1}(b_1, b_2) = \emptyset$ and $i \geq 7$, it follows from Lemma 4.1 that $|L(b_1)| = |R(b_2)| = 3$. Therefore, since $G$ is cubic, there are at least four consecutive out-vertices in $C_{i-1}(b_1', b_2')$. Since $C_{i-2}(c_1, c_2) = \emptyset$ and since $i - 1 \geq 6$, it follows from Lemma 4.1 that $|L(c_1)| = |R(c_2)| = 3$. Hence $|R(b_1')| = |B(b_1)| = |B(b_2)| \geq 10$.

We claim that $|B(b_1)| = |B(b_2)| \geq 12$. For otherwise, $b_1'$ is adjacent to both $L(b_1)$ and $L(c_1)$, or $b_2'$ is adjacent to both $R(b_2)$ and $R(c_2)$. By symmetry, we may assume the former. Then since $G$ is cubic, $|L(b_1')| \geq 5 \leq |A(b_1')|$. Since $|R(b_1')| \geq 10$, $\ell(b_1') \geq (5, 5, 10)$ and $b_1'$ is non-positive, a contradiction.

Then $|B(c_1)| = |B(c_2)| < 12$; for otherwise, $\ell(c_1) \geq (3, 12, 12)$ and $c_1$ is non-positive, a contradiction.

![Figure 9](image-url)
Since $i - 2 \geq 5$, it follows from Lemma 4.1 that $|L(d_1)| = |R(d_2)| = 3$ or $|C_{i-3}(d_1, d_2)| \neq 0$. So by Lemma 4.2, $|L(d_1)| = |R(d_2)| = 3$ or $|C_{i-3}(d_1, d_2)| \geq 2$. Thus $|B(c_1)| = |B(c_2)| \geq 10$. Since $|B(c_1)| = |B(c_2)| < 12$, either $c_1'$ is adjacent to $L(c_1)$, or $c_2'$ is adjacent to $R(c_2)$. By symmetry, we may assume the former. Then $|A(c_1')| \geq 5$. So $|L(c_1')| = 4$, or else $\ell(c_1') \geq (5, 5, 10)$ and $c_1'$ would be non-positive. Hence $|A(c_1')| \geq 6$. Since $|R(c_1')| \geq 10$, $|A(c_1')| = 6$ (or else, $\ell(c_1') \neq (4, 7, 10)$ and $c_1'$ would be non-positive). So $G$ has an edge $xy$ such that $x \in V(C_{i-2})$, $y$ is strictly between $C_{i-1}$ and $C_{i-2}$, and $y$ is adjacent to $L(c_1)$. See Figure 9. Since $G$ is cubic, we can check that $\ell(y) \geq (4, 6, 12)$ and $y$ is non-positive, a contradiction.

**Lemma 4.4.** Let $i \geq 8$, and let $a_1$ and $a_2$ be consecutive in-vertices on $C_i$ such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$. Let $b_1$ and $b_2$ be the out-vertices on $C_{i-1}$ such that $R(b_1) = L(b_2) = R(a_1)$, and assume $|C_{i-1}(b_1, b_2)| \geq 3$. Let $b_3, b_4$ be the vertices on $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$. Then

1. both $b_3$ and $b_4$ are out-vertices on $C_{i-1}$,
2. $|L(b_1)| = |R(b_2)| = 3$ and $|R(b_3)| = |L(b_4)| = 3$, and
3. both $R(b_3)$ and $L(b_4)$ use three consecutive vertices on $C_{i-1}$.

**Proof.** Since $|C_i(a_1, a_2)| \geq 4$ and $|C_{i-1}(b_1, b_2)| \geq 3$, $|R(b_1)| = |L(b_2)| \geq 10$.

(1) Suppose $b_3$ is an in-vertex on $C_{i-1}$. See Figure 10. Then $|A(b_3)| = |R(b_1)| \geq 11$, $|L(b_3)| = |B(b_1)| \geq 5$, and $|R(b_3)| \geq 4$. Hence, $|R(b_3)| = 4$; otherwise, $\ell(b_3) \geq (5, 5, 11)$ and $b_3$ would be non-positive. So let $b, c, c_3$ be the vertices of $R(b_3)$ such that $b \in C_{i-1}$, $\{c, c_3\} \subset V(C_{i-2})$, $c$ is adjacent to $b$, and $c_3$ is adjacent to $b_3$.

Since $|B(b_1)| \geq 5$, $|L(b_1)| \leq 4$; for otherwise, $\ell(b_1) \geq (5, 5, 11)$ and $b_1$ would be non-positive. In fact $|L(b_1)| = 4$, for otherwise, $|A(b_3)| = |R(b_1)| \geq 12$ and $|L(b_3)| = |B(b_1)| \geq 6$, and hence, $\ell(b_3) \geq (4, 6, 12)$ and $b_3$ would be non-positive. Therefore, by Lemma 3.3, we see that $|L(b_1) \cap C_{i-1}| \in \{2, 3\}$. Hence we have two cases to consider.

First, assume that $|L(b_1) \cap C_{i-1}| = 2$. See Figure 10(a). Then $|B(b_1)| \geq 6$. Hence $|R(b_1)| = 11$ and $|B(b_1)| = 6$, for otherwise, $\ell(b_1) \geq (4, 6, 12)$ or

![Figure 10](image-url)
(4, 7, 11), and $b_1$ would be non-positive. Then $|C_{i-1}(b_1, b_2)| = 3$, and $b_4$ is also an in-vertex. Hence, $|R(b_4)| \geq 5$, $|L(b_4)| \geq 4$, and $|A(b_4)| = 11$. If $|L(b_4)| \geq 5$, then $\ell(b_4) \geq (5, 5, 11)$ and $b_4$ would be non-positive. So $|L(b_4)| = 4$. Thus $b_4$ is adjacent to a vertex $c_4$ on $C_{i-2}$ and $C_{i-2}(c_3, c) = \emptyset = C_{i-2}(c, c_4)$. Since $|L(b_2)| = 11$, $a_2$ is adjacent to $b_2$, and hence $|R(b_2)| \geq 4$. Moreover, $|R(b_2)| = 4$ and $|B(b_2)| = 6$; otherwise, $\ell(b_2) \geq (5, 5, 11)$ and $b_2$ would be non-positive.

So $|B(b_2)| \geq 6$, for otherwise, $\ell(b_2) \geq (4, 7, 11)$ and $b_2$ would be non-positive. Let $c_1, c_2$ be the out-vertices on $C_{i-2}$ such that $R(c_1) = B(b_1)$ and $L(c_2) = B(b_2)$. See Figure 10(a). Since $|B(b_1)| = |B(b_2)| = 6$, $C_{i-2}(c_1, c_3) = \emptyset = C_{i-2}(c_3, c_2)$. So $c_1 \neq c_2$ (since $i - 2 \geq 5$ and $G$ is 2-connected). Let $c_1', c_2'$ be the in-vertices on $C_{i-2}$ such that $R(c_1') = L(c_2') = B(b_1) = B(b_2)$. Then $|C_{i-2}(c_1', c_2')| \geq 5$. Since $i - 2 \geq 5$, it follows from Lemma 4.1 and (1) of Lemma 4.2 that $|B(c_1)| = |B(c_2)| \geq 11$. So $|L(c_1)| \leq 4$, or else, $\ell(c_1) \geq (5, 6, 11)$ and $c_1$ would be non-positive. Since $|R(c_1)| = 6$, we have $|L(c_1)| = 4$ and $|L(c_1) \cap C_{i-2}| = 2$. Therefore, $|B(c_1)| \geq 12$ and $\ell(c_1) \geq (4, 6, 12)$, and so, $c_1$ is non-positive, a contradiction.

So $|L(b_1) \cap C_{i-1}| = 3$. See Figure 10(b) and (c). Then $|R(b_1)| \geq 12$ (since $|C_{i-1}(b_1, b_2)| \geq 3$ and both $b_2$ and $b$ are in-vertices). So $|B(b_1)| = 5$, otherwise $\ell(b_2) \geq (4, 6, 12)$ and $b_3$ would be non-positive. Let $c_1$ denote the out-vertex on $C_{i-2}$, such that $R(c_1) = B(b_1)$. Since $|B(b_1)| = 5$ and $|L(b_1)| = 4$, $|L(c_1)| \geq 5$ and $C_{i-2}(c_1, c_3) = \emptyset$. Hence $|B(c_1)| = |B(c)| \geq 7$. Note that $|L(c_1)| \in \{5, 6\}$, otherwise $\ell(c_1) \geq (5, 7, 7)$ and $c_1$ would be non-positive. Let $d_1, d_2$ be the out-vertices on $C_{i-3}$ such that $R(d_1) = L(d_2) = B(c_1)$. Assume $|L(c_1)| = 6$. See Figure 10(b). Then $|B(c_1)| = |R(d_1)| = 7$, or else, $\ell(c_1) \geq (5, 6, 8)$ and $c_1$ would be non-positive. Thus $C_{i-3}(d_1, d_2) = \emptyset$, and so, $|B(d_1)| \geq 6$. So $|L(d_1)| = 5$, or $\ell(d_1) \geq (6, 6, 7)$ and $d_1$ would be non-positive. Therefore, $|L(d_1) \cap C_{i-3}| = 2$ and $|B(d_1)| \geq 7$. Hence $\ell(d_1) \geq (5, 7, 7)$ and $d_1$ is non-positive, a contradiction.

Now assume $|L(c_1)| = 5$. See Figure 10(c). Then $|L(c_1) \cap C_{i-2}| = 2$. Let $c_1', c_2'$ denote the in-vertices on $C_{i-2}$ such that $R(c_1') = L(c_2') = B(c_1)$. Then $|C_{i-2}(c_1', c_2')| \geq 4$. Since $i - 2 \geq 5$, it follows from Lemma 4.1 and (1) of Lemma 4.2 that $|B(c_1)| \geq 10$. Thus $\ell(c_1) \geq (5, 5, 10)$ and $c_1$ is non-positive, a contradiction.

Similarly, we can prove that $b_4$ is an out-vertex.

(2) By symmetry, we only prove (2) for $b_3$ and $b_1$. By (1), $b_3$ is an out-vertex, and so, $|L(b_3)| = |R(b_1)| \geq 10$ and $|B(b_1)| = |B(b_3)| \geq 6$. Hence $|L(b_1)| \leq 4$; for otherwise, $\ell(b_1) \geq (5, 6, 10)$ and $b_1$ would be non-positive.

First, assume that $|L(b_1)| = 4$. Then $|L(b_1) \cap C_{i-1}| = 2$, then $|B(b_1)| \geq 7$ and $\ell(b_1) \geq (4, 7, 10)$, and hence $b_1$ is non-positive, a contradiction. So $|L(b_1) \cap C_{i-1}| = 3$. Then $|R(b_1)| \geq 11$. Further, $|R(b_1)| = 11$ and $|B(b_1)| = 6$; otherwise, $\ell(b_1) \geq (4, 6, 12)$ or $\ell(b_1) \geq (4, 7, 11)$, and $b_1$ would be non-positive. Because $|R(b_1)| = 11$, $b_3$ is adjacent to $b_3$ and $a_2$ is adjacent to $b_2$. See Figure 11(a). So $|B(b_2)| \geq 6$ and $|R(b_2)| \geq 4$, and if $|R(b_2)| = 4$ then $|B(b_2)| \geq 7$. Thus $\ell(b_2) \geq (4, 7, 11)$ or $(5, 6, 11)$, and $b_2$ is non-positive, a contradiction.
Lemma 4.1 that it follows from (4.2) that Lemma 4.6.

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Therefore, \(|L(b_1)| = 3\). So \(|B(b_1)| = |B(b_3)| \geq 7\) and \(|L(b_3)| = |R(b_1)| \geq 11\). Therefore, \(|R(b_3)| = 3\), or \(b_3\) would be non-positive with \(\ell(b_3) \geq (4, 7, 11)\).

(3) By symmetry, we will only show that \(R(b_3)\) uses three consecutive vertices on \(C_{i-1}\). Suppose on the contrary that \(R(b_3)\) contains a vertex, say \(b\), not on \(C_{i-1}\). See Figure 11(b). Let \(b'\) denote the vertex in \(R(b_3) - \{b, b_3\}\). Note that \(C_{i-1}(b, b') = 0\) and \(|R(b_1)| \geq 13\) (by (2)). Let \(b', b^*\) denote the in-vertices on \(C_{i-1}\) such that \(R(b'_1) = L(b^*) = B(b_1)\). Note that \(|C_{i-1}(b'_1, b^*)| \geq 4\).

Then \(b* \in C_{i-1}(b', b_2)\). For otherwise, \(|C_{i-1}(b'_1, b^*)| \geq 7\), and it follows from Lemma 4.1 that \(|B(b_1)| \geq 12\). So \(\ell(b_1) \geq (3, 12, 13)\) and \(b_1\) is non-positive, a contradiction. Therefore, since \(b_4\) is an out-vertex and \(|L(b_4)| = 3\) (by (2)), \(b\) is not adjacent to \(b_4\), and so, \(|R(b_1)| = |L(b_2)| \geq 14\). Hence, \(|B(b_1)| \leq 10\), or else, \(\ell(b_1) \geq (3, 11, 14)\) and \(b_1\) would be non-positive.

Let \(c_1, c^*\) denote the out-vertices on \(C_{i-2}\) such that \(R(c_1) = L(c^*) = B(b_1)\). Since \(|C_{i-2}(b'_1, b^*)| \geq 4\) and \(i - 1 \geq 7\), it follows from Lemma 4.1 that \(|L(c_1)| = |R(c^*)| = 3\) or \(|C_{i-2}(c_1, c^*)| \neq 0\). Moreover, if \(|C_{i-2}(c_1, c^*)| \neq 0\), then it follows from (4.2) that \(|C_{i-2}(c_1, c^*)| \geq 3\). Therefore \(|B(b_1)| \geq 10\). Since \(|B(b_1)| \leq 10\), we have \(|B(b_1)| = 10\). So \(b'_i\) is adjacent to \(L(b_i)\). Hence \(|A(b'_i)| \geq 5\). Then \(|L(b'_1)| = 4\) (or else \(\ell(b'_1) \geq (5, 5, 10)\) and \(b'_i\) would be non-positive), and so, \(|A(b'_1)| = 6\) (or else \(\ell(b'_1) \geq (4, 7, 10)\) and \(b'_1\) would be non-positive). So \(a_i\) is not adjacent to \(L(b_1)\), and therefore, \(|R(b_1)| \geq 15\). But then \(\ell(b_1) \geq (3, 10, 15)\) and \(b_1\) is non-positive, a contradiction.

Lemma 4.5. Let \(i \geq 8\), and let \(a_1\) and \(a_2\) be consecutive in-vertices on \(C_i\) such that (i) \(R(a_1) = L(a_2)\) and (ii) \(|C_i(a_1, a_2)| \geq 4\). Let \(b_1\) and \(b_2\) be the out-vertices on \(C_{i-1}\) such that \(R(b_1) = L(b_2) = R(a_1)\). Then \(|L(b_1)| = |R(b_2)| = 3\).

Proof. \(C_{i-1}(b_1, b_2) = \emptyset\), then Lemma 4.5 follows from Lemma 4.1. If \(C_{i-1}(b_1, b_2) \neq \emptyset\), then by Lemma 4.2, \(|C_{i-1}(b_1, b_2)| \geq 3\). Therefore, Lemma 4.5 follows from Lemma 4.4.

Lemma 4.6. Let \(i \geq 9\), and let \(a_1\) and \(a_2\) be consecutive in-vertices on \(C_i\) such that (i) \(R(a_1) = L(a_2)\) and (ii) \(|C_i(a_1, a_2)| \geq 4\). Let \(b_1\) and \(b_2\) be the out-vertices on \(C_{i-1}\) such that \(R(b_1) = L(b_2) = R(a_1)\), and assume that \(|C_{i-1}(b_1, b_2)| \geq 3\). Let \(b_3, b_4\) be the vertices on \(C_{i-1}(b_1, b_2)\) such that \(C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)\).
Let $b, b^*$ be the vertices in $C_{i-1}(b_3, b_4)$ such that $C_{i-1}(b_3, b) = \emptyset = C_{i-1}(b^*, b_4)$, and let $b'$ be the neighbor of $b$ not on $C_{i-1}$ and let $b''$ be the neighbor of $b^*$ not on $C_{i-1}$. Then

1. $b, b^*$ are in-vertices and $b', b'' \notin C_{i-2}$, and
2. $b'$ is contained in a facial triangle of $G$ which also contains two consecutive vertices on $C_{i-2}$, and $b''$ is contained in a facial triangle of $G$ which also contains two consecutive vertices on $C_{i-2}$.

**Proof.** By Lemma 4.4, $|L(b_1)| = |R(b_2)| = 3$, $R(b_3) = A(b)$, and $L(b_4) = A(b^*)$ are facial triangles of $G$, and $b$ and $b^*$ are in-vertices on $C_{i-1}$. By symmetry, we only need to prove (1) and (2) for $b'$.

1. Suppose $b' \in C_{i-2}$. See Figure 12(a). Then $b'$ is an out-vertex on $C_{i-2}$. Hence $|B(b')| \geq 5$, $|R(b')| \geq 5$, and $|L(b')| = |B(b_1)| \geq 7$. If $|B(b')| = 5$, then $|L(b')| \geq 8$ and $|R(b')| \geq 6$, and hence, $\ell(b') \geq (5, 6, 8)$ and $b'$ would be non-positive. So $|B(b')| \geq 6$. Then $|R(b')| = 5$ or else $\ell(b') \geq (6, 7)$ and $b'$ would be non-positive. Therefore, $|R(b') \cap C_{i-2}| = 2$, and so, if $|B(b')| = 6$ then $|L(b')| \geq 8$. So $\ell(b') \geq (5, 6, 8)$ or $(5, 7, 7)$, and hence, $b'$ is non-positive, a contradiction.

2. First we show that $b'$ is contained in a facial triangle of $G$. Suppose on the contrary that $b'$ is not contained in any facial triangle. Since $b' \notin C_{i-2}$, $|L(b)| \geq 8$. Hence $|R(b)| \leq 7$, otherwise, $\ell(b') \geq (4, 8, 8)$ and $b'$ would be non-positive. Thus $b \neq b^*$, and so, $|R(b_1)| \geq 14$. See Figure 12(b).

Let $x$ denote the vertex on $C_{i-1}(b, b^*) - A(b)$ such that $x$ is adjacent to $A(b)$. Then $x$ is an out-vertex; for otherwise, $|L(x)| \geq 6$ and $|R(x)| \geq 4$, and so, $\ell(x) \geq (4, 6, 14)$ and $x$ would be non-positive. Thus $|R(b)| = 7$. This implies that $R(x)$ is a triangle (otherwise $\ell(x) \geq (4, 7, 14)$ and $x$ would be non-positive) and $R(x)$ uses three consecutive vertices of $C_{i-1}$. Now let $c$ denote the out-vertex on $C_{i-2}$ such that $L(c) = R(b)$. Then since $|L(c)| = 7$, $c$ is adjacent to $C_{i-1}$ and $|L(c) \cap C_{i-2}| = 2$. Hence $|R(c)| \geq 5$ and $|B(c)| \geq 6$. Furthermore, if $|R(c)| = 5$ then $|B(c)| \geq 7$. So $\ell(c) \geq (5, 7, 7)$ or $(6, 6, 7)$ and $c$ is non-positive, a contradiction.

Next we show that the facial triangle of $G$ containing $b'$ also contains two consecutive vertices on $C_{i-2}$. Note that $|R(b_1)| = |L(b_2)| \geq 12$ and $|B(b_1)| \geq 8 \leq |B(b_2)|$ (because $b', b'' \notin C_{i-2}$ by (1)). Also note that $|B(b_1)| \leq 11 \geq |B(b_2)|$, as

![Figure 12](image-url)
otherwise, there exists \( i \in \{1, 2\} \) such that \( \ell(b_i) \geq (3, 12, 12) \) and \( b_i \) is non-positive, a contradiction.

Let \( b'_1, b'_2 \) denote the in-vertices on \( C_{i-1} \) such that \( R(b'_1) = B(b_1) \) and \( L(b'_2) = B(b_2) \). Let \( c_1, c_2, c_3, c_4 \) be the out-vertices on \( C_{i-2} \) such that \( R(c_1) = L(c_3) = B(b_1) \) and \( L(c_2) = R(c_4) = B(b_2) \). See Figure 13.

For convenience, let \( P \) denote the clockwise subpath of \( B(b_1) \) from \( b' \) to \( c_3 \). We show that \( |P| = 2 \), and therefore, since \( G \) is cubic, the facial triangle of \( G \) containing \( b' \) also contains two consecutive vertices on \( C_{i-2} \).

Suppose \( |P| \geq 4 \). Then \( |B(b_1)| \geq 10 \). Recall that \( |B(b_1)| \leq 11 \). First, assume that \( C_{i-2}(c_1, c_3) \neq \emptyset \). Then \( |B(b_1)| = 11 \), and \( C_{i-2}(c_1, c_3) \) consists of only one vertex, say \( c \). It is easy to see that \( \ell(c) \geq (5, 5, 11) \) and \( c \) is non-positive, a contradiction. So \( C_{i-2}(c_1, c_3) = \emptyset \). Thus \( |B(b_1)| \geq 6 \). So \( |L(c_1)| \leq 4 \), otherwise, \( \ell(c_1) \geq (5, 6, 10) \) and \( c_1 \) would be non-positive. If \( |L(c_1)| = 4 \) and \( |L(c_1) \cap C_{i-2}| = 2 \), then \( |B(c_1)| \geq 7 \) and \( \ell(c_1) \geq (4, 7, 10) \), and hence, \( c_1 \) is non-positive, a contradiction. So \( |L(c_1)| = 3 \) or \( |L(c_1)| = 4 \) and \( |L(c_1) \cap C_{i-2}| = 3 \). Then \( c_1 \) is not adjacent to \( b' \). So \( |B(b_1)| = 11 \), and hence, \( b'_1 \) is adjacent to \( L(b_1) \).

Therefore \( |A(b'_1)| \geq 5 \) \( |L(b'_1)| \), and so, \( \ell(b'_1) \geq (5, 5, 11) \) and \( b'_1 \) is non-positive, a contradiction.

Now assume that \( |P| = 3 \). Note that \( 9 \leq |B(b_1)| \leq 11 \). Also note that \( c_1 \neq c_4 \), and so, \( c_3 \notin L(c_1) \). Therefore, it follow from Lemma 4.5 that \( |C_{i-1}(b'_1, b)| \leq 3 \). So \( b'_1 \) is adjacent to \( L(b_1) \), and hence, \( |A(b'_1)| \geq 5 \).

Assume \( |A(b'_1)| = 5 \). See Figure 13(a). Then \( |L(b'_1)| = 5 \) and \( |R(b'_1)| = |B(b_1)| = 9 \), for otherwise, \( \ell(b'_1) \geq (5, 6, 9) \) or \( (5, 5, 10) \) and \( b'_1 \) would be non-positive. Hence \( C_{i-2}(c_1, c_3) = \emptyset \) and \( |B(c_1)| \geq 7 \). Therefore, \( \ell(c_1) \geq (5, 7, 9) \) and \( b_1 \) is non-positive, a contradiction.

So \( |A(b'_1)| \geq 6 \). Then \( |L(b'_1)| = 4 \), or else, \( \ell(b'_1) \geq (5, 6, 9) \) and \( b'_1 \) would be non-positive. Thus \( b'_1 \) is adjacent to \( c_1 \) and \( |L(b'_1) \cap C_{i-1}| = |L(b'_1) \cap C_{i-2}| = 2 \).

See Figure 13(b). Assume \( C_{i-2}(c_1, c_3) = \emptyset \). Then \( |B(c_1)| \geq 7 \). In fact, \( |B(c_1)| = 7 \), for otherwise, \( \ell(c_1) \geq (4, 8, 9) \) and \( c_1 \) would be non-positive. So \( |R(c_3)| \geq 5 \) and \( \ell(c_3) \geq (5, 7, 9) \), and hence, \( c_3 \) is non-positive, a contradiction. Thus \( C_{i-2}(c_1, c_3) \neq \emptyset \), and so, \( |B(b_1)| \geq 10 \). This implies that \( |A(b'_1)| = 6 \), or else, \( \ell(b'_1) \geq (4, 7, 10) \) and \( b'_1 \) would be non-positive. Thus \( A(b'_1) \) has an edge \( xy \) such as:

![Figure 13](image-url)
Lemma 4.7. Let $i \geq 10$, and let $a_1$ and $a_2$ be consecutive in-vertices on $C_i$ such that
(i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| = 4$. Let $b_1$ and $b_2$ be the out-vertices on $C_{i-1}$ such that $R(b_1) = L(b_2) = R(a_1)$, and assume that $|C_{i-1}(b_1, b_2)| \geq 3$. Then $|C_{i-1}(b_1, b_2)| = 3$.

Proof. See Figure 14. Let $b_3, b_4$ be the vertices on $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$. Let $b, b^*$ be the vertices in $C_{i-1}(b_3, b_4)$ such that $C_{i-1}(b, b) = \emptyset = C_{i-1}(b^*, b_4)$. By Lemma 4.6, $b$ and $b^*$ are in-vertices on $C_{i-1}$.

By Lemma 4.4, $|A(b)| = |A(b^*)| = 3 = |L(b_1)| = |R(b_2)|$, and $A(b)$ and $A(b^*)$ each contain three consecutive vertices on $C_{i-1}$. So if $b = b^*$ then $|C_{i-1}(b_1, b_2)| = 3$. Hence we may assume that $b \neq b^*$.

Let $b'$ be the neighbor of $b$ not on $C_{i-1}$ and let $b''$ be the neighbor of $b^*$ not on $C_{i-1}$. Since $i \geq 10$, it follows from Lemma 4.6 that $b', b'' \notin C_{i-2}$, there is a facial triangle containing $b'$ and two consecutive vertices on $C_{i-2}$, and there is a facial triangle containing $b''$ and two consecutive vertices on $C_{i-2}$. See Figure 14(a).

Since $G$ is cubic, $A(b) \cap A(b^*) = \emptyset$. So $|R(b_1)| = |L(b_2)| \geq 14$. Also $|B(b_1)| \geq 8 \leq |B(b_2)|$ (since $b', b'' \notin C_{i-2}$), and $|B(b_1)| \leq 10 \geq |B(b_2)|$ (otherwise, there would exist $i \in \{1, 2\}$ such that $\ell(b_i) \geq (3, 11, 14)$ and $b_i$ is non-positive).

Let $b_1', b_2'$ be the in-vertices on $C_{i-1}$ such that $R(b_1') = B(b_1)$ and $L(b_2') = B(b_2)$, and let $c_1, c_2, c_3, c_4$ be the out-vertices on $C_{i-2}$ such that $R(c_1) = L(c_3) = B(b_1)$ and $R(c_4) = L(c_2) = B(b_2)$. Then $c_3$ is adjacent to $b'$, $c_4$ is adjacent to $b''$, and $|R(c_3)| = |L(c_4)| = 3$.

\[ \text{FIGURE 14. Proof of Lemma 4.7.} \]
Case 1. $b'_1$ is adjacent to $L(b_1)$ or $b'_2$ is adjacent to $R(b_2)$.

By symmetry, we may assume that $b'_1$ is adjacent to $L(b_1)$. Thus $|A(b'_1)| \geq 5$. See Figure 14(b).

We claim that $|B(b_1)| \leq 9$. For otherwise, $|B(b_1)| = 10$. So $|R(b_1)| = 14$, or else, $\ell(b_1) \geq (3, 10, 15)$ and $b_1$ would be non-positive. Then $a_1$ is adjacent to $L(b_1)$, and so, $|A(b'_1)| \geq 6$. So $|L(b'_1)| = 4$; otherwise, $\ell(b'_1) \geq (5, 6, 10)$ and $b'_1$ would be non-positive. Hence, $|L(b'_1) \cap C_{i-1}| = 2$ and $|A(b'_1)| \geq 7$. Therefore, $\ell(b'_1) \geq (4, 7, 10)$ and $b'_1$ is non-positive, a contradiction.

Since $|B(b_1)| \leq 9$, $|C_{i-2}(c_1, c_3)| \leq 1$. Suppose $|C_{i-2}(c_1, c_3)| = 1$. Let $c$ denote the only vertex in $C_{i-2}(c_1, c_3)$. Then $c$ is an in-vertex, $|A(c)| = |B(b_1)| = 9$ and $|L(c)| \geq 5$. Since $|R(c_3)| = 3$, $|R(c_3)| \geq 6$. Thus $\ell(c) \geq (5, 6, 9)$ and $c$ is non-positive, a contradiction. Therefore $|C_{i-2}(c_1, c_3)| = 0$, and hence $|B(c_1)| \geq 7$. So $|L(c_1)| \leq 4$, for otherwise, $\ell(c_1) \geq (5, 7, 8)$ and $c_1$ would be non-positive.

If $|L(c_1)| = 4$ and $|L(c_1) \cap C_{i-2}| = 2$, then $|B(c_1)| \geq 8$ and $c_1$ is non-positive with $\ell(c_1) \geq (4, 8, 8)$, a contradiction. So $|L(c_1)| = 3$ or $|L(c_1)| = 4$ and $|L(c_1) \cap C_{i-2}| = 3$. Then $|R(b'_1)| = |R(c_1)| = 9$ and $|L(b'_1)| \geq 5$. Further, if $|L(b'_1)| = 5$, then $|A(b'_1)| \geq 6$. Thus $b'_1$ is non-positive with $\ell(b'_1) \geq (5, 6, 9)$, a contradiction.

Case 2. $b'_1$ is not adjacent to $L(b_1)$ and $b'_2$ is not adjacent to $R(b_2)$.

Then $|C_{i-1}(b'_1, b)| \geq 4 \leq |C_{i-1}(b'_1, b'_2)|$. Thus, since $i-1 \geq 9$ and by Lemma 4.5, $|L(c_1)| = |R(c_2)| = 3$. Hence $|B(b_1)| \geq 10 \leq |B(b_2)|$. In fact, $|B(b_1)| = 10 = |B(b_2)|$ and $|R(b_1)| = 14 = |L(b_2)|$; for otherwise, $\ell(b_1) \geq (3, 11, 14)$ or $(3, 10, 15)$, and so, $b_1$ would be non-positive. So there are exactly six vertices in $C_{i-1}(b_1, b_2)$, and $C_{i-2}(c_1, c_3) = \emptyset = C_{i-2}(c_4, c_2)$.

Let $c'_1$ and $c'$ denote the in-vertices on $C_{i-2}$ such that $R(c'_1) = L(c') = B(c_1)$. We claim that $c'_1 \in C_{i-2}(c_3, c_4)$. Otherwise, $c' \in C_{i-2}(c_2, c'_1)$ and $|C_{i-2}(c'_1, c')| \geq 8$. Since $i-2 \geq 8$, it follows from Lemma 4.5 that $|R(c'_1)| \geq 14$. Thus $|R(c'_1)| = 14 = |B(c_1)|$, or else $\ell(c_1) \geq (3, 10, 15)$ and $c_1$ would be non-positive. Then $c'_1$ is adjacent to $L(c_1)$ and $|A(c'_1)| \geq 6$. Therefore $\ell(c'_1) \geq (4, 6, 14)$ and $c'_1$ is non-positive, a contradiction.

So $c' \in C_{i-2}(c_3, c_4)$. See Figure 14(c). If $c'$ is adjacent to $R(c_3)$ or $L(c_4)$, then it is easy to check that $\ell(c') \geq (4, 8, 8)$ or $(5, 7, 8)$, and $c'$ is non-positive, a contradiction.

So $c'$ is adjacent to neither $R(c_3)$ nor $L(c_4)$.

Let $x$ be the vertex in $C_{i-2}(c_3, c_4) - V(R(c_3))$ such that $x$ is adjacent to $R(c_3)$. By the choice of $c'$, $x$ is an out-vertex on $C_{i-2}$ and $|L(x)| = |R(b)| \geq 10$. Further, $|L(x)| = |R(b)| = 10$, or else $\ell(b) \geq (3, 11, 14)$ and $b$ would be non-positive. Also $|R(x)| = 3$, as otherwise $\ell(x) \geq (4, 10, 10)$ and $x$ would be non-positive. So $C_{i-2}(x, c') = \emptyset$ and $A(c') = R(x)$.

Let $y$ denote the vertex in $C_{i-2}(c', c_4) - V(L(c_4))$ such that $y$ is adjacent to $L(c_4)$. By the same argument as above for $x$, we can show that $C_{i-2}(c', y) = \emptyset$ and $A(c') = L(y)$.
Let $c'_i$ be the in-vertex on $C_{i-2}$ such that $L(c'_i) = B(c_2)$. Let $d_1, d_2, d_3, d_4$ be the out-vertices on $C_{i-3}$ such that $R(d_1) = L(d_3) = B(c_1) = B(c_3)$ and $R(d_4) = L(d_2) = B(c_2) = B(c_4)$. Since $C_{i-2}(c_1, c_3) = \emptyset$ and $|C_{i-1}(b'_i, b)| \geq 4$, it follows from Lemma 4.3 that $C_{i-3}(d_1, d_3) \neq \emptyset$. Since $C_{i-2}(c_4, c_2) = \emptyset$ and $C_{i-1}(b^*, b'_2) \geq 4$, it follows from Lemma 4.3 that $C_{i-3}(d_4, d_2) \neq \emptyset$. Since $i - 2 \geq 8$ and by Lemma 4.5, $|L(d_1)| = |R(d_3)| = 3 = |L(d_4)| = |R(d_2)|$. So $|L(c')| \geq 12 \leq |R(c')|$. Thus $\ell(c') \geq (3, 12, 12)$ and $c'$ is non-positive, a contradiction.

Lemma 4.8. Let $i \geq 11$, and let $a_1$ and $a_2$ be consecutive in-vertices on $C_i$ such that $(i) \ R(a_1) = L(a_2)$ and $(ii) \ |C_i(a_1, a_2)| \geq 4$. Let $b_1, b_2$ denote the out-vertices on $C_{i-1}$ such that $R(b_1) = L(b_2) = R(a_1)$. Then $|C_{i-1}(b_1, b_2)| = 0$.

**Proof.** Suppose on the contrary that $|C_{i-1}(b_1, b_2)| \neq 0$. Since $i - 1 \geq 10$, it follows from Lemma 4.2 that $|C_{i-1}(b_1, b_2)| \geq 3$. Therefore, by Lemma 4.7, $|C_{i-1}(b_1, b_2)| = 3$. Let $b_3, b_4$ be the vertices on $C_{i-1}(b_1, b_2)$ in that clockwise order from $b_1$ to $b_2$. See Figure 15. By Lemma 4.4, $|L(b_1)| = |R(b_2)| = 3$ and $|A(b)| = |R(b_2)| = |L(b_4)| = 3$. Let $b' \in$ be the neighbor of $b$ not on $C_{i-1}$. By Lemma 4.6, $b' \notin C_{i-2}$ and there is a facial triangle of $G$ containing $b'$ and two consecutive vertices on $C_{i-2}$. Let $b'_1, b'_2$ denote the in-vertices on $C_{i-1}$ such that $R(b'_1) = B(b_1)$ and $L(b'_2) = B(b_2)$. Let $c_1, c_2, c_3, c_4$ denote the out-vertices on $C_{i-2}$ such that $R(c_1) = L(c_3) = B(b_1)$ and $R(c_2) = L(c_4) = B(b_2)$.

Note that $|L(b_1)| = |L(b_2)| \geq 12$ and $|B(b_1)| = 8 = |B(b_2)|$. So $|B(b_1)| \leq 11 = |B(b_2)|$, for otherwise, there exists $i \in \{1, 2\}$ such that $\ell(b_i) \geq (3, 12, 12)$ and $b_i$ is non-positive, a contradiction.

**Case 1.** $b'_i$ is adjacent to $L(b_1)$, or $b'_2$ is adjacent to $R(b_2)$.

By symmetry we may assume that $b'_1$ is adjacent to $L(b_1)$. Then $|A(b'_1)| \geq 5$.

First, assume $|A(b'_1)| = 5$. See Figure 15(a). Then $|A(b'_1) \cap C_{i-1}| = 3$ and $|L(b'_1)| \geq 5$. In fact $|L(b'_1)| = 5$ and $|L(b'_1) \cap C_{i-2}| = 2$, as otherwise, $\ell(b'_1) \geq (5, 6, 8)$ and $b'_1$ would be non-positive. So $b'_1$ is adjacent to $c_1$ and $|B(c_1)| \geq 6$. Therefore, $\ell(c_1) \geq (5, 6, 8)$ and $c_1$ is non-positive, a contradiction.

So $|A(b'_1)| \geq 6$. Then $|L(b'_1)| \leq 4$, or else $\ell(b'_1) \geq (5, 6, 8)$ and $b'_1$ would be non-positive. In fact $|L(b'_1)| = 4$ (since $G$ is cubic), $b'_1$ is adjacent to $c_1$, and $|L(b'_1) \cap C_{i-2}| = 2$. See Figure 15(b). Then $C_{i-2}(c_1, c_3) \neq \emptyset$; otherwise, $|B(c_1)| \geq 6$. Then $|L(b'_1)| \leq 4$. Therefore, $\ell(b'_1) \geq (5, 6, 8)$ and $b'_1$ would be non-positive. In fact $|L(b'_1)| = 4$ (since $G$ is cubic), $b'_1$ is adjacent to $c_1$, and $|L(b'_1) \cap C_{i-2}| = 2$. See Figure 15(b).

![Figure 15. Proof of Lemma 4.8.](image-url)
8 and $\ell(c_1) \geq (4, 8, 8)$, and so, $c_1$ is non-positive, a contradiction. Suppose $C_{i-2}(c_1, c_3)$ consists of only one vertex, say $c$. Then $|L(c)| \geq 6$, $|R(c)| \geq 6$, and $|A(c)| \geq 9$. So $\ell(c) \geq (6, 6, 9)$ and $c$ is non-positive, a contradiction. Hence, $|C_{i-2}(c_1, c_3)| \geq 2$. Then $|B(b_1)| \geq 10$. So $|A(b'_1)| = 6$; otherwise, $\ell(b'_1) \geq (4, 7, 10)$ and $b'_1$ would be non-positive. Thus $A(b'_1)$ has an edge $xy$ such that $x \in C_{i-1}$, $y \in R(a_1) \cap A(b'_1)$, and $x, y \notin L(b_1)$. Now it is easy to see that $\ell(y) \geq (4, 6, 12)$ and $y$ is non-positive, a contradiction.

Case 2. $b'_1$ is not adjacent to $L(b_1)$, and $b'_2$ is not adjacent to $R(b_2)$.

Then $|C_{i-1}(b'_1, b)| \geq 4$ and $|C_{i-1}(b, b'_2)| \geq 4$. Since $i - 1 \geq 10$, it follows from Lemma 4.5 that $|L(c_1)| = |R(c_3)| = |L(c_4)| = |R(c_2)| = 3$. Thus $|B(b_1)| \geq 10 \leq |B(b_2)|$ and $|B(c_1)| \geq 6 \leq |B(c_2)|$.

We claim that $C_{i-2}(c_1, c_3) = \emptyset = C_{i-2}(c_4, c_2)$. For otherwise, we may assume by symmetry that $C_{i-2}(c_1, c_3) \neq \emptyset$. Then $|B(b_1)| \geq 11$. Thus $|B(b_1)| = 11$ and $C_{i-2}(c_1, c_3)$ consists of only one vertex, say $c$. Now $\ell(c) \geq (6, 6, 11)$ and $c$ is non-positive, a contradiction.

Let $c'_1, c'_2$ denote the in-vertices on $C_{i-2}$ such that $R(c'_1) = L(c'_2) = B(c_1)$, and let $d_1, d_2$ denote the out-vertices on $C_{i-3}$ such that $R(d_1) = L(d_2) = B(c_1)$. See Figure 15(c).

Since $i - 2 \geq 9$ and $|C_{i-2}(c'_1, c'_2)| \geq 6$, it follows from Lemma 4.5 that $|L(d_1)| = |R(d_2)| = 3$. Thus $|B(c_1)| \geq 12$. Since $|C_{i-1}(b'_1, b)| \geq 4$ and $C_{i-2}(c_1, c_3) = \emptyset$, it follows from Lemma 4.3 that $|C_{i-3}(d_1, d_2)| \neq 0$. So by Lemma 4.2, $|C_{i-3}(d_1, d_2)| \geq 3$. Therefore, $|B(c_1)| \geq 14$. In fact $|B(c_1)| = 14$, or else $\ell(c_1) \geq (3, 10, 15)$ and $c_1$ would be non-positive. So $c'_1$ is adjacent to $L(c_1)$. Then $|A(c'_1)| \geq 5$. Since $|L(d_1)| = 3$, $|L(c'_1)| \geq 5$. So $\ell(c'_1) \geq (5, 5, 14)$ and $c'_1$ is non-positive, a contradiction.

We are now ready to prove our main lemma in this section.

**Lemma 4.9.** Let $i \geq 12$, let $a_1$ and $a_2$ be consecutive in-vertices on $C_i$ such that $R(a_1) = L(a_2)$. Then $|C_i(a_1, a_2)| \leq 3$.

**Proof.** Let $b_1, b_2$ be the out-vertices on $C_{i-1}$ such that $R(b_1) = R(a_1) = L(b_2)$. Suppose for a contradiction that $|C_i(a_1, a_2)| \geq 4$. Then by Lemma 4.8, $|C_{i-1}(b_1, b_2)| = 0$. So by Lemma 4.5, $|L(b_1)| = |R(b_2)| = 3$. Let $b'_1, b'_2$ be the in-vertices on $C_{i-1}$ such that $R(b'_1) = L(b'_2) = B(b_1)$, and let $c_1, c_2$ be the out-vertices on $C_{i-2}$ such that $R(c_1) = L(c_2) = B(b_1)$. See Figure 16.

![Figure 16](image-url)
Then $|C_{i-1}(b_1', b_2')| \geq 4$. Since $i - 1 \geq 11$, it follows from Lemma 4.3 that $|C_{i-2}(c_1, c_2)| \neq 0$. On the other hand, since $i - 1 \geq 11$, it follows from Lemma 4.8 that $|C_{i-2}(c_1, c_2)| = 0$, a contradiction. \hfill \Box

5. THREE VERTICES BETWEEN CONSECUTIVE IN-VERTICES

Assume that $G$ is a positively curved, cubic, infinite plane graph, which is nicely embedded in the plane with respect to a nice sequence $(C_0, C_1, \ldots)$. In this section, we show that, for sufficiently large $i$, there are at most two vertices between any two consecutive in-vertices on $C_i$. As in Section 4, this is done through a series of lemmas.

**Lemma 5.1.** Let $i \geq 15$, and let $a_1$ and $a_2$ be consecutive in-vertices on $C_i$ such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| = 3$. Let $b_1, b_2$ denote the out-vertices on $C_{i-1}$ such that $R(b_1) = L(b_2) = R(a_1)$. Then $|C_{i-1}(b_1, b_2)| \geq 2$.

**Proof.** Suppose on the contrary that $|C_{i-1}(b_1, b_2)| \leq 1$. Let $b_1', b_2'$ denote the in-vertices on $C_{i-1}$ such that $R(b_1') = B(b_1)$ and $B(b_2) = L(b_2')$. Let $c_1, c_2$ be the out-vertices on $C_{i-2}$ such that $R(c_1) = R(b_1')$ and $L(c_2) = L(b_2')$. See Figure 17.

**Case 1.** $|C_{i-1}(b_1, b_2)| = 1$.

Let $b$ be the only vertex in $C_{i-1}(b_1, b_2)$. See Figure 17(a). Since $G$ is 2-connected, $b$ is an in-vertex. Note that $|A(b)| \geq 8$ and $|L(b)| \geq 5 \leq |R(b)|$. In fact, $|L(b)| = |R(b)| = 5$, or else $\ell(b) \geq (5, 6, 8)$ and $b$ would be non-positive. So $C_{i-1}(b_1', b_1) = \emptyset = C_{i-1}(b_2, b_2'), \ |C_{i-2}(c_1, c_2)| = 1$, and both $b_1'$ and $b_2'$ are adjacent to $C_i$. Hence, $|B(c_1)| \geq 7$.

Moreover, $a_1$ is adjacent to $b_1$, or $a_2$ is adjacent to $b_2$; for otherwise, $|R(b_1)| \geq 10$, and so, $\ell(b) \geq (5, 5, 10)$ and $b_1$ is non-positive, a contradiction. By symmetry, we may assume that $a_1$ is adjacent to $b_1$. Then $|L(b_1)| \geq 5$. Furthermore, $|L(b_1)| = 5$, as otherwise, $\ell(b_1) \geq (5, 6, 8)$ and $b_1$ would be non-positive. So $|L(b_1) \cap C_{i-1}| = 3$. This implies that $|L(c_1)| = |L(b_1')| \geq 5$.

We claim that $|L(c_1)| = 6$ and $|L(c_1) \cap C_{i-2}| = 3$. If $|L(c_1) \cap C_{i-2}| = 2$, then there are four consecutive out-vertices on $C_{i-2}$, contradicting Lemma 4.9. So $|L(c_1) \cap C_{i-2}| \geq 3$. Then $|L(c_1)| \geq 6$. In fact, $|L(c_1)| = 6$; otherwise, $\ell(c_1) \geq 5$.

![FIGURE 17. Proof of Lemma 5.1.](image-url)
(5, 7, 7) and \(c_1\) would be non-positive. Since \(|L(b_1) \cap C_{i-1}| = 3\) and \(|L(c_1) \cap C_{i-2}| \geq 3\), we have \(|L(c_1) \cap C_{i-2}| = 3\).

So \(|B(c_1)| = 7\), or else \(\ell(c_1) \geq (5, 6, 8)\) and \(c_1\) would be non-positive. Now let \(d_1, d_2\) denote the out-vertices on \(C_{i-3}\) such that \(R(d_1) = L(d_2) = B(c_1)\).

Since \(|B(c_1)| = 7 = |R(d_1)|, C_{i-3}(d_1, d_2) = \emptyset\). So \(|B(d_1)| \geq 6\). Since \(|L(c_1) \cap C_{i-2}| = 3, |L(d_1)| \geq 5,\) and \(|L(d_1)| = 5\) implies \(|B(d_1)| \geq 7\). So \(\ell(d_1) \geq (5, 7, 7)\) or \((6, 6, 7)\), and hence, \(d_1\) is non-positive, a contradiction.

**Case 2.** \(|C_{i-1}(b_1, b_2)| = 0\).

By Lemma 4.9, \(C_{i-1}(b_1', b_1) = \emptyset\) or \(C_{i-1}(b_2, b_2') = \emptyset\). By symmetry, we may assume \(C_{i-1}(b_1', b_1) = \emptyset\). So \(|L(b_1)| \geq 4, |R(b_1)| \geq 7, |B(b_1)| \geq 6\). See Figure 17(b).

**Claim 5.1.** \(|L(b_1)| = 4\) and \(|L(b_1) \cap C_{i-1}| = 3\).

Suppose \(|L(b_1)| \geq 5\). Then \(|L(b_1)| = 5\); otherwise, \(\ell(b_1) \geq (6, 6, 7)\) and \(b_1\) would be non-positive. Moreover, \(a_2\) is adjacent to \(b_2\) and \(a_1\) is adjacent to \(b_1\), for otherwise \(\ell(b_1) \geq (5, 6, 8)\) and \(b_1\) would be non-positive. So \(|L(b_1) \cap C_{i-1}| = 3\). Also \(|B(b_1)| = 6\), or else \(\ell(b_1) \geq (5, 7, 7)\) and \(b_1\) would be non-positive. So \(C_{i-2}(c_1, c_2) = \emptyset\); \(c_1\) is adjacent to \(b_1'\), \(c_2\) is adjacent to \(b_2\), and \(C_{i-1}(b_2, b_2') = \emptyset\).

Thus \(|R(c_1)| = 6\) and \(|B(c_1)| \geq 6\). Hence \(|L(c_1)| = 5\), as otherwise \(\ell(c_1) \geq (6, 6, 6)\) and \(c_1\) would be non-positive. Therefore, \(|L(c_1) \cap C_{i-2}| = 2\).

Since \(C_{i-1}(b_2, b_2') = \emptyset\) and because \(a_2\) is adjacent to \(b_2\), \(|R(b_2)| \geq 5\). So by a symmetric argument as above, we have \(|R(b_2)| = 5, |R(c_2)| = 5,\) and \(|R(c_2) \cap C_{i-2}| = 2\). Then \(C_{i-2}\) has four distinct consecutive out-vertices, contradicting Lemma 4.9.

So \(|L(b_1)| = 4\). Therefore, since \(|C_{i-1}(b_1', b_1)| = 0, |L(b_1) \cap C_{i-1}| = 3\).

**Claim 5.2.** \(|R(b_2)| = 4\) and \(|R(b_2) \cap C_{i-1}| = 3\).

Since \(|L(b_1)| = 4\) and \(|L(b_1) \cap C_{i-1}| = 3, |R(b_1)| \geq 8\) and \(|L(c_1)| \geq 5\).

Assume \(C_{i-1}(b_2, b_2') = \emptyset\). Then \(|B(b_1)| = 7\). In fact, \(|B(b_1)| = 7;\) otherwise, \(\ell(b_1) \geq (4, 8, 8)\) and \(b_1\) would be non-positive. So \(C_{i-2}(c_1, c_2) = \emptyset\) and \(c_1\) is adjacent to \(b_1'\). Therefore, \(|B(c_1)| \geq 6, |L(c_1)| \geq 5,\) and \(|R(c_1)| = |B(b_1)| = 7\).

Moreover, if \(|L(c_1)| = 5\) then \(|B(c_1)| \geq 7\). Hence \(\ell(c_1) \geq (5, 7, 7)\) or \((6, 6, 7)\) and \(c_1\) is non-positive, a contradiction.

Thus \(C_{i-1}(b_2, b_2') = \emptyset\). Therefore \(|R(b_2)| \geq 4\). In fact \(|R(b_2)| = 4,\) otherwise \(\ell(b_2) \geq (5, 6, 8)\) and \(b_2\) would be non-positive. So \(|R(b_2) \cap C_{i-1}| = 3\).

By Claims 5.1 and 5.2, \(|R(b_1)| = |L(b_2)| \geq 9, |L(c_1)| \geq 5,\) and \(|R(c_2)| \geq 5\). See Figure 17(c).

**Claim 5.3.** \(|C_{i-2}(c_1, c_2)| = 1\).

Suppose \(|C_{i-2}(c_1, c_2)| = 0\). Then \(|B(c_2)| = |B(c_1)| \geq 6\). Hence \(|R(c_2)| = |L(c_1)| = 5,\) for otherwise, there would exist \(i \in \{1, 2\}\) such that \(c_i\) is non-positive with \(\ell(c_i) \geq (6, 6, 6)\). So \(|L(c_1) \cap C_{i-2}| = 2 = |R(c_2) \cap C_{i-2}|\). Then there
are four consecutive out-vertices on $C_{i-2}$, contradicting Lemma 4.9. So $C_{i-2}(c_1, c_2) \neq \emptyset$. Then $|C_{i-2}(c_1, c_2)| = 1$; otherwise $|B(b_1)| \geq 8$ and $\ell(b_1) \geq (4, 8, 9)$, and so, $b_1$ would be non-positive.

By Claim 5.3, let $c$ denote the only vertex in $C_{i-2}(c_1, c_2)$. Then $|L(c)| \geq 5 \leq |R(c)|$. Moreover, $|L(c)| = 5$ or $|R(c)| = 5$, for otherwise, $\ell(c) \geq (6, 6, 6)$ and $c$ would be non-positive. By symmetry, we may assume that $|L(c)| = 5$. So $c$ is adjacent to a vertex on $C_{i-3}$, say $d$. Then $|L(c_1)| \geq 5$. In fact $|L(c_1)| = 6$, or $\ell(c_1) \geq (5, 7, 7)$ and $c_1$ would be non-positive. Also, $|R(c)| \in \{5, 6\}$, for otherwise, $\ell(c) \geq (5, 7, 7)$ and $c$ would be non-positive. Let $d_1, d_2$ denote the out-vertices on $C_{i-3}$ such that $R(d_1) = B(c_1)$ and $L(d_2) = B(c_2)$. Because $|L(c)| = 5$, $C_{i-3}(d_1, d) = \emptyset$.

We claim that $C_{i-3}(d_1, d, 0)$. For otherwise, $|B(c_2)| = |R(c)| = 6$ (since $|R(c)| \in \{5, 6\}$). So $|R(c_2)| \geq 6$. Then $\ell(c_2) \geq (6, 6, 7)$ and $c_2$ is non-positive, a contradiction.

Hence $d_1, d_2$ are consecutive out-vertices on $C_{i-3}$. Let $d_1', d_2'$ denote the in-vertices on $C_{i-3}$ such that $R(d_1') = B(d_1) = B(d_2) = L(d_2')$. Let $e_1$ be the out-vertex on $C_{i-4}$ such that $R(e_1) = B(d_1)$. Then, since $i - 3 \geq 12$ and by Lemma 4.9, $C_{i-3}(d_1', d_1) = 0 = C_{i-3}(d_2, d_2')$. So $|B(d_1)| = |R(d_1')| \geq 7$ and $|A(d_1')| \geq 6$. Hence $|L(d_1')| = 5$, or else $\ell(d_1') \geq (6, 6, 7)$ and $d_1'$ would be non-positive. Therefore, $d_1'$ is adjacent to $e_1$, $|L(e_1)| = 5$, and $|B(e_1)| \geq 6$. Since $|L(e_1)| = 5$, either $|B(e_1)| \geq 7$ or $|R(e_1)| \geq 8$. So $\ell(e_1) \geq (5, 7, 7)$ or $(5, 6, 8)$, and $e_1$ is non-positive, a contradiction.

**Lemma 5.2.** Let $i \geq 15$, and let $a_1$ and $a_2$ be consecutive in-vertices on $C_i$ such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| = 3$. Let $b_1, b_2$ denote the out-vertices on $C_{i-1}$ such that $R(b_1) = L(b_2) = R(a_1)$. Then $|C_{i-1}(b_1, b_2)| \geq 3$.

**Proof.** Suppose on the contrary that $|C_{i-1}(b_1, b_2)| \leq 2$. By Lemma 5.1, $C_{i-1}(b_1, b_2)$ has exactly two vertices, say $b_3$ and $b_4$. See Figure 18. Without loss of generality, we may assume that $b_3 \in C_{i-1}(b_1, b_4)$. Since $G$ is 2-connected and by Lemma 3.4, both $b_3$ and $b_4$ are in-vertices. So $|R(b_1)| = |L(b_2)| \geq 9$. Note that $|L(b_3)| = |B(b_1)| \geq 5 \leq |B(b_2)| = |R(b_4)|$. So $|R(b_3)| = |L(b_4)| \leq 5$, for otherwise, $\ell(b_3) \geq (5, 6, 9)$ and $b_3$ would be non-positive.

**Case 1.** $|R(b_3)| = |L(b_4)| = 5$.

Then $|B(b_1)| = |B(b_2)| = 5$, for otherwise, there exists $i \in \{1, 2\}$ such that $\ell(b_i) \geq (5, 6, 9)$ and $b_i$ is non-positive, a contradiction. Therefore $|R(b_3)| \cap C_{i-2} = 3$. Let $c$ be the vertex in $R(b_3) \cap C_{i-2}$ not adjacent to $b_3$ or $b_4$. Then $c$ is an in-vertex on $C_{i-2}$ and $|L(c)| \geq 6 \leq |R(c)|$. See Figure 18(a). Note that $c$ has a neighbor, say $d$, on $C_{i-3}$; for otherwise, $\ell(c) \geq (5, 7, 7)$ and $c$ would be non-positive. Also note that $|B(d)| \geq 5$, and if $|B(d)| = 5$ then $|L(d)| \geq 7 \leq |R(d)|$. So $\ell(d) \geq (5, 7, 7)$ or $(6, 6, 6)$, and $d$ is non-positive, a contradiction.

**Case 2.** $|R(b_3)| = |L(b_4)| = 4$. 

Let $c_3, c_4 \in C_{i-2}$ be the neighbors of $b_3, b_4$, respectively. Let $c_1, c_2$ be the out-vertices on $C_{i-2}$ such that $R(c_1) = B(b_1)$ and $L(c_2) = B(b_2)$. See Figures 18(b) and (c).

We claim that $|C_{i-2}(c_1, c_3)| \leq 1 \geq |C_{i-2}(c_4, c_2)|$. For otherwise, we may assume by symmetry that $|C_{i-2}(c_1, c_3)| \geq 2$. Then $|L(b_3)| \geq 7$. In fact, $|L(b_3)| = 7$; otherwise, $\ell(b_3) \geq (4, 8, 9)$ and $b_3$ would be non-positive. So $b_1'$ is adjacent to $b_1$ and $|L(b_1)| \geq 4$. Moreover, if $|L(b_1)| = 4$ then $|R(b_1)| \geq 10$. Therefore, $\ell(b_1) \geq (5, 7, 9)$ or $(4, 7, 10)$, and $b_1$ is non-positive, a contradiction.

We further claim that $C_{i-2}(c_1, c_3) \cup C_{i-2}(c_4, c_2) \neq \emptyset$. For otherwise, $C_{i-2}(c_1, c_3) \cup C_{i-2}(c_4, c_2) = \emptyset$. Since $G$ is cubic and 2-connected, $c_1 \neq c_2$ and $|C_{i-2}(c_2, c_1)| \neq \emptyset$. Therefore, $C_{i-2}$ has four consecutive out-vertices, contradicting Lemma 4.9.

Figure 18(c). Then $|C_{i-2}(c_4, c_2)| = 1$. Let $c^*$ denote the only vertex in $C_{i-2}(c_4, c_2)$. See Figure 18(c). Then $c^*$ is adjacent to some vertex $d^*$ on $C_{i-3}$; otherwise, $\ell(c^*) \geq (6, 6, 6)$ and $c^*$ would be non-positive. Note that $|R(c)| = |L(c^*)| \geq 6$.

So $|L(c)| = 5 = |R(c^*)|$; for otherwise, $\ell(c) \geq (6, 6, 6)$ and $c$ would be non-positive or $\ell(c^*) \geq (6, 6, 6)$ and $c^*$ would be positive. Therefore, $|R(c)| = |L(c^*)| \leq 7$, as otherwise, $\ell(c) \geq (5, 6, 8)$ and $c$ would be non-positive. Then, since $i - 3 \geq 12$ and by Lemma 4.9, $C_{i-3}(d, d^*) \neq \emptyset$. So $|C_{i-3}(d, d^*)| = 1$ (because $|R(c)| \leq 7$). Let $d'$ denote the only vertex in $C_{i-3}(d, d^*)$. It is easy to see that $\ell(d') \geq (6, 6, 7)$, and so, $d'$ is non-positive, a contradiction.

**Lemma 5.3.** Let $i \geq 17$, and let $a_1$ and $a_2$ be consecutive in-vertices on $C_i$ such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| = 3$. Let $b_1, b_2$ denote the out-vertices on $C_{i-1}$ such that $R(b_1) = L(b_2) = R(a_1)$, and let $b_3, b_4$ be the vertex in $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$. Then both $b_3$ and $b_4$ are in-vertices.
Proof. Note that \( b_3 \) and \( b_4 \) are well defined by Lemma 5.2. Suppose this lemma is false. By symmetry, we may assume that \( b_3 \) is an out-vertex. See Figure 19. Then \(|B(b_1)| = |B(b_3)| \geq 6\). By Lemma 5.2, \( |C_{i-1}(b_1, b_2)| \geq 3\), and so, \(|R(b_1)| = |L(b_2)| \geq 9\). Hence \(|L(b_1)| \leq 4\), for otherwise, \( \ell(b_1) \geq (5, 6, 9) \) and \( b_1 \) would be non-positive. Let \( b_1', b \) be the in-vertices on \( C_{i-1} \) such that \( R(b_1') = L(b) = B(b_1) \), and let \( c_1, c \) be the out-vertices on \( C_{i-2} \) such that \( R(c_1) = L(c) = B(b_1) \). Note that \( b \in C_{i-1}(b_3, b_4) \), as otherwise, \( \{b_3, b_4\} \) is a 2-cut in \( G \) that is contained in \( V(C_{i-1}) \), contradicting Lemma 3.4. Since \(|L(b_1)| \leq 4\), we have two cases to consider.

**Case 1.** \(|L(b_1)| = 3\).

Then \(|R(b_1)| \geq 10\). Since \( i - 1 \geq 16 \) and \(|C_{i-1}(b_1', b)| \geq 3\), it follows from Lemma 4.9 that \(|C_{i-1}(b_1', b)| = 3\). Hence, \( b_1' \) is adjacent to \( L(b_1) \) and \( C_{i-1}(b_3, b) = \emptyset \). So \(|A(b_1')| \geq 5\). By Lemma 5.2, \(|C_{i-2}(c_1, c)| \geq 3\), and so, \(|B(b_1)| \geq 9\). Therefore, \(|R(b_3)| = 3\), as otherwise \( \ell(b_3) \geq (4, 9, 10) \) and \( b_3 \) would be non-positive. So \( V(R(b_3)) \subseteq V(C_{i-1}) \). See Figure 19(a).

Note that \( b \) is not adjacent to \( c \). For otherwise, \(|L(c)| \geq 9\) and \(|R(c)| \geq 5 \leq |B(c)|\). Further, if \(|B(c)| = 5\) then \(|R(c)| \geq 6\). So \( \ell(c) \geq (5, 6, 9) \) and \( c \) is non-positive, a contradiction.

Hence, \(|L(c)| = |R(b_1')| \geq 10\). Then \(|L(b_1')| = 4\), as otherwise, \( \ell(b_1') \geq (5, 5, 10) \) and \( b_1' \) would be non-positive. Therefore \(|A(b_1')| \geq 6\). In fact, \(|A(b_1')| = 6\); otherwise, \( \ell(b_1') \geq (4, 7, 10) \) and \( b_1' \) is non-positive, a contradiction. So \( R(a_1) \cap A(b_1') \neq \emptyset \) and \(|R(a_1)| \geq 11\). Let \( y \) be the vertex in \( R(a_1) \cap A(b_1') \) such that the clockwise path in \( R(a_1) \) from \( y \) to \( a_1 \) is shortest. Let \( C \) denote the facial cycle of \( G \) containing \( y \) such that \( C \neq R(a_1) \) and \( C \neq A(b_1') \). Note that \(|C| \geq 4\). If \(|C| \geq 5\), then \( \ell(y) \geq (5, 6, 11) \) and \( y \) is non-positive, a contradiction. So \(|C| = 4\). Then \(|R(a_1)| \geq 12\), and so, \( \ell(y) \geq (4, 6, 12) \) and \( y \) is non-positive, a contradiction.

**Case 2.** \(|L(b_1)| = 4\).

First, assume that \(|L(b_1) \cap C_{i-1}| = 2\). See Figure 19(b). Then \(|C_{i-1}(b_1', b)| \geq 3\). By Lemma 4.9, \(|C_{i-1}(b_1', b)| = 3\). Since \( i - 1 \geq 16 \) and by Lemma 5.2, \(|C_{i-2}(c_1, c)| \geq 3\). So \(|B(b_1)| \geq 9\). Hence, \( \ell(b_1) \geq (4, 9, 9) \) and \( b_1 \) is non-positive, a contradiction.

![Figure 19](image-url)
So \(|L(b_1) \cap C_{i-1}| = 3\). See Figure 19(c). Then \(|R(b_1)| \geq 10\). Thus \(|B(b_1)| = 6\), as otherwise, \(\ell(b_1) \geq (4, 7, 10)\) and \(b_1\) would be non-positive. So \(b\) is adjacent to \(c\), \(b'_1\) is adjacent to \(c_1\), \(|C_{i-1}(b'_1, b)| = 2\), and \(|C_{i-2}(c_1, c)| = \emptyset\). Note that \(|B(c_1)| \geq 6\). Because \(|L(b_1) \cap C_{i-1}| = 3\) and \(|L(b_1)| = 4\), \(|L(b'_1)| = |L(c_1)| \geq 5\).

In fact, \(|L(c_1)| = 5\) and \(|L(c_1) \cap C_{i-2}| = 2\), as otherwise, \(\ell(c_1) \geq (6, 6, 6)\) and \(c_1\) would be non-positive. Let \(c'\) be the vertex on \(C_{i-2}\) such that \(R(c') = L(c_1)\). Then \(c', c_1\) and \(c\) are three consecutive out-vertices on \(C_{i-2}\). Since \(i - 2 \geq 15\) and by Lemma 5.2, \(|B(c_1)| \geq 9\). Thus \(\ell(c_1) \geq (5, 6, 9)\) and \(c_1\) is non-positive, a contradiction. \(\blacksquare\)

**Lemma 5.4.** Let \(i \geq 17\), and let \(a_1\) and \(a_2\) be consecutive in-vertices on \(C_i\) such that \(R(a_1) = L(a_2)\). Then \(|C_i(a_1, a_2)| \leq 2\).

**Proof.** Suppose this lemma is false. Then by Lemma 4.9, \(|C_i(a_1, a_2)| = 3\). Let \(b_1, b_2\) denote the out-vertices on \(C_{i-1}\) such that \(R(b_1) = L(b_2) = R(a_1)\). By Lemma 5.2, \(|C_{i-1}(b_1, b_2)| \geq 3\). Let \(b_3, b_4\) be the vertices in \(C_{i-1}(b_1, b_2)\) such that \(C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)\). By Lemma 5.3, both \(b_3\) and \(b_4\) are in-vertices. See Figure 20. So \(|R(b_1)| = |L(b_2)| \geq 10\) and \(|B(b_1)| = |L(b_3)| \geq 5 \leq |R(b_4)| = |B(b_2)|\).

Then \(|R(b_3)| = 4 = |L(b_4)|\); for otherwise, there exists \(i \in \{3, 4\}\) such that \(\ell(b_i) \geq (5, 5, 10)\) and \(b_i\) is non-positive, a contradiction. Also \(|L(b_1)| \leq 4 \geq |R(b_2)|\); otherwise, there exists \(i \in \{1, 2\}\) such that \(\ell(b_i) \geq (5, 5, 10)\) and \(b_i\) is non-positive, a contradiction.

Let \(b'_1\) be the in-vertex on \(C_{i-1}\) such that \(R(b'_1) = B(b_1)\). Let \(c_1, c_3, c_4\) be out-vertices on \(C_{i-2}\) such that \(R(c_1) = L(c_3) = B(b_1)\) and \(R(c_4) = B(b_2)\). Since \(|R(b_3)| = |L(b_4)| = 4\), \(c_3\) is adjacent to \(b_3\), and \(c_4\) is adjacent to \(b_4\). Let \(c'_1, c^*\) be the in-vertices on \(C_{i-2}\) such that \(R(c'_1) = L(c^*) = B(c_1)\).

We claim that \(|L(b_1)| = 4 = |R(b_2)|\). For otherwise, we may assume by symmetry that \(|L(b_1)| = 3\). See Figure 20(a). Then \(|R(b_1)| \geq 11\) and \(|B(b_1)| \geq 6\). Indeed, \(|R(b_1)| = 11\) and \(|B(b_1)| = 6\), as otherwise, \(\ell(b_3) \geq (4, 7, 11)\) or \((4, 6, 12)\), and \(b_3\) would be non-positive. So \(|C_{i-1}(b_1, b_2)| = 3\). Since \(|R(b_3)| = |L(b_4)| = 4\), \(C_{i-2}[c_3, c_4]\) consists of three consecutive out-vertices on \(C_{i-2}\). Since \(i - 2 \geq 15\) and by Lemma 4.9, \(c_1\) and \(c_3\) cannot be adjacent. Thus \(|B(b_1)| \geq 7\), a contradiction.

![Figure 20](image-url)
We further claim that \(|L(b_1) \cap C_{i-1}| = 3 = |R(b_2) \cap C_{i-1}|\). For otherwise, we may assume by symmetry that \(|L(b_1) \cap C_{i-1}| = 2\). See Figure 20(b). Then \(|B(b_1)| \geq 6\). Indeed \(|B(b_1)| = 6\); otherwise \(\ell(b_1) \geq (4, 7, 10)\) and \(b_1\) would be non-positive. Thus \(b_3\) is adjacent to \(c_3\), \(b'_1\) is adjacent to \(c_1\), and \(C_{i-2}(c_1, c_3) = \emptyset\). Thus \(|C_{i-2}(c', c^*)| \geq 3\) (because \(|R(b_3)| = 4\)). By Lemma 4.9, \(C_{i-2}(c', 1) = \emptyset\), and so \(|L(c_1)| \geq 6\). By Lemma 5.2, \(|B(c_1)| \geq 9\). Since \(|R(c_1)| = |B(b_1)| = 6\, \ell(c_1) \geq (5, 6, 9)\) and \(c_1\) is non-positive, a contradiction.

So \(|R(b_1)| = |L(b_2)| \geq 12\). See Figure 20(c). Then \(|L(b_3)| = 5\), for otherwise, \(\ell(b_3) \geq (4, 6, 12)\) and \(b_3\) would be non-positive. Thus \(b'_1\) is adjacent to \(c_1\), and \(C_{i-2}(c_1, c_3) = \emptyset\). Again, \(|C_{i-2}(c', c^*)| \geq 3\) (because \(|R(b_3)| = 4\)). By Lemma 4.9, \(C_{i-2}(c', 1) = \emptyset\), and so \(|L(c_1)| \geq 6\). By Lemma 5.2, \(|B(c_1)| \geq 9\). Therefore \(\ell(c_1) \geq (5, 6, 9)\) and \(c_1\) is non-positive, a contradiction.

\section{Two Vertices Between Consecutive In-Vertices}

Let \(G\) be a positively curved, cubic, infinite, plane graph that is nicely embedded with respect to a nice sequence \((C_0, C_1, \ldots)\). In this section, we show that, for sufficiently large \(i\), there is at most one vertex between any two consecutive in-vertices on \(C_i\). Again, this is done through a series of lemmas.

\begin{lemma}
Let \(i \geq 20\), and let \(a_1\) and \(a_2\) be consecutive in-vertices on \(C_i\) such that (i) \(R(a_1) = L(a_2)\) and (ii) \(|C_i(a_1, a_2)| = 2\). Let \(b_1, b_2\) denote the out-vertices on \(C_{i-1}\) such that \(R(b_1) = L(b_2) = R(a_1)\). Then \(|C_{i-1}(b_1, b_2)| \neq 0\).

\begin{proof}
Suppose \(|C_{i-1}(b_1, b_2)| = 0\). Then \(|B(b_1)| = |B(b_2)| \geq 6\). Let \(b'_1, b'_2\) be the in-vertices on \(C_{i-1}\) such that \(R(b'_1) = B(b_1) = L(b'_2)\), and let \(c_1, c_2\) denote the out-vertices on \(C_{i-2}\) such that \(R(c_1) = L(c_2) = B(b_1)\). Because \(i - 1 \geq 19\) and by Lemma 5.4, \(|C_{i-1}(b'_1, b'_2)| = 2\). Hence \(C_{i-1}(b'_1, b_1) = \emptyset = C_{i-1}(b_2, b'_2)\). So \(|L(b_1)| \geq 4 \leq |R(b_2)|\).

\textbf{Case 1.} \(|C_{i-2}(c_1, c_2)| = 0\).

Let \(c'_1, c'_2\) denote the in-vertices on \(C_{i-2}\) such that \(R(c'_1) = B(c_1) = L(c'_2)\). See Figure 21(a). Since \(i - 2 \geq 18\) and by Lemma 5.4, \(|C_{i-2}(c'_1, c'_2)| = 2\). Thus \(C_{i-2}(c'_1, c_1) = \emptyset = C_{i-2}(c_2, c'_2)\). Note that \(|B(c_2)| \geq 6\). Then \(|L(c_1)| = |R(c_2)| = 5\);\end{proof}

\end{lemma}
otherwise, there exists $i \in \{1, 2\}$ such that $\ell(c_i) \geq (6, 6, 6)$ and $c_i$ is non-positive, a contradiction. This forces $|L(b_1)| \geq 5 \leq |R(b_2)|$. Further $|L(b_1)| = 5 = |R(b_2)|$, for otherwise, there exists $i \in \{1, 2\}$ such that $\ell(b_i) \geq (6, 6, 6)$ and $b_i$ is non-positive, a contradiction. Therefore, $|R(b_1)| = |L(b_2)| \geq 8$. But then $\ell(b_1) \geq (5, 6, 8)$ and $b_1$ is non-positive, a contradiction.

**Case 2.** $|C_{i-2}(c_1, c_2)| = 1$.

Let $c$ be the only vertex in $C_{i-2}(c_1, c_2)$. See Figure 21(b). Then $|A(c)| \geq 7$ and $|L(c)| \geq 5 \leq |R(c)|$. If $|L(c) \cap C_{i-3}| = 2 = |R(c) \cap C_{i-3}|$, then $C_{i-3}$ has three consecutive out-vertices, contradicting Lemma 5.4 (because $i - 3 \geq 17$). So by symmetry, we may assume that $|R(c) \cap C_{i-3}| \geq 3$. Hence $|R(c)| \geq 6$. Indeed $|R(c)| = 6$ and $|L(c)| = 5$, for otherwise, $\ell(c) \geq (5, 7, 7)$ or $(6, 6, 7)$ and $c$ would be non-positive. So $|R(c) \cap C_{i-2}| = 3$, and hence, $|R(c_2)| \geq 5$. In fact, $|R(c_2)| = 5$, as otherwise, $\ell(c_2) \geq (6, 6, 7)$ and $c_2$ would be non-positive. Therefore $|R(b_2)| \geq 5$. Further, if $|R(b_2)| = 5$ then $|L(b_2)| \geq 7$. So $\ell(b_2) \geq (5, 7, 7)$ or $(6, 6, 7)$, and $c_2$ is non-positive, a contradiction.

**Case 3.** $|C_{i-2}(c_1, c_2)| \geq 2$.

Then $|B(b_1)| = |B(b_2)| \geq 8$. Hence $|L(b_1)| = |R(b_2)| = 4$; otherwise there exists $i \in \{1, 2\}$ such that $\ell(b_i) \geq (5, 6, 8)$ and $b_i$ is non-positive, a contradiction. Since $C_{i-1}(b_1', b_1) = \emptyset = C_{i-1}(b_2, b_2')$, $|L(b_1) \cap C_{i-1}| = |R(b_2) \cap C_{i-1}|$. Hence, $|R(b_1)| \geq 8$ and $\ell(b_1) \geq (4, 8, 8)$, and so, $b_1$ is non-positive, a contradiction. ■

**Lemma 6.2.** Let $i \geq 21$, and let $a_1$ and $a_2$ be consecutive in-vertices on $C_i$ such that (i) $R(a_1) = L(a_2)$ and (iii) $|C_i(a_1, a_2)| = 2$. Let $b_1, b_2$ denote the out-vertices on $C_{i-1}$ such that $R(b_1) = L(b_2) = R(a_1)$. Then $|C_{i-1}(b_1, b_2)| \neq 1$.

**Proof.** Suppose $|C_{i-1}(b_1, b_2)| = 1$. Then $|R(b_1)| = |L(b_2)| \geq 7$. Let $b$ be the only vertex in $C_{i-1}(b_1, b_2)$. Since $G$ is 2-connected, $b$ is an in-vertex. See Figure 22. Note that $|L(b)| = |B(b_1)| \geq 5 \leq |B(b_2)| = |R(b)|$.

Now $|L(b)| = 5$ or $|R(b)| = 5$, as otherwise, $\ell(b) \geq (6, 6, 7)$ and $b$ would be non-positive. By symmetry, we may assume that $|L(b)| = 5$. Then $b$ is adjacent to a vertex on $C_{i-3}$, say $c$. Let $c_1$ denote the out-vertex on $C_{i-2}$ such that $R(c_1) = B(b_1)$, and let $b_1'$ be the in-vertex on $C_{i-1}$ such that $R(b_1') = B(b_1)$.

Since $|L(b)| = 5$, $C_{i-2}(c_1, c) = \emptyset = C_{i-1}(b_1', b_1)$, and $b_1'$ is adjacent to $c_1$.

![Figure 22. Proof of Lemma 6.2.](image-url)
Let $c'_1, c'$ denote the in-vertices on $C_{i-2}$ such that $R(c'_1) = L(c') = B(c_1)$. Since $i - 2 \geq 19$ and by Lemma 5.4, $|C_{i-2}(c'_1, c')| = 2$. Hence $C_{i-2}(c, c') = \emptyset = C_{i-2}(c_1, c_1)$. Thus $|A(c')| \geq 6$. Indeed, $|A(c')| = 6$, as otherwise $\ell(b) \geq (5, 7, 7)$ and $b$ would be non-positive. Now $|R(c')| = 5$ and $|L(c')| \leq 7$, for otherwise, $\ell(c') \geq (5, 6, 8)$ or $(6, 6, 6)$ and $c'$ would be non-positive.

Let $d_1, d_2$ denote the out-vertices on $C_{i-3}$ such that $R(d_1) = L(d_2) = B(c_1)$. Since $|R(c')| = 5$ and $|A(c')| = 6$, $c'$ is adjacent to $d_2$ and $|R(c') \cap C_{i-3}| = 2$. So $|B(d_2)| \geq 6$. Moreover, since $i - 3 \geq 18$ and by Lemma 5.4, $C_{i-3}(d_1, d_2) \neq \emptyset$. Therefore, since $|L(c')| \leq 7$, $|L(c')| = 7$. So let $d$ be the only vertex in $C_{i-3}(d_1, d_2)$. Then $|R(d)| = |B(d_2)| \geq 6$, and $|L(d)| = |B(d_1)| \geq 5$. In fact, $|R(d)| = 5$ and $|R(d)| = 6$, for otherwise, $\ell(d) \geq (6, 6, 7)$ or $(5, 7, 7)$ and $d$ would be non-negative. Thus $|L(d) \cap C_{i-4}| = 2 = |R(d) \cap C_{i-4}|$. Hence $C_{i-4}$ has three consecutive out-vertices. Since $i - 4 \geq 17$, this contradicts Lemma 5.4.

Lemma 6.3. Let $i \geq 24$, and let $a_1$ and $a_2$ be consecutive in-vertices on $C_i$ such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| = 2$. Let $b_1, b_2$ denote the out-vertices on $C_{i-1}$ such that $R(b_1) = L(b_2) = R(a_1)$, and assume that $|C_{i-1}(b_1, b_2)| \geq 2$. Let $b_3, b_4$ be the vertices in $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$. Then both $b_3$ and $b_4$ are out-vertices.

Proof. Suppose this is false. Without loss of generality, we may assume that $b_3$ is an in-vertex. See Figure 23. Since $|C_{i-3}(b_1, b_2)| \geq 2$, $|A(b_3)| = |R(b_1)| = |L(b_2)| \geq 8$. Hence $b_3$ is adjacent to $C_{i-2}$; otherwise, $|L(b_3)| \geq 6$ and $|R(b_3)| \geq 5$, and so, $\ell(b_3) \geq (5, 6, 8)$ and $b_3$ is non-positive, a contradiction. Let $c$ denote the neighbor of $b_3$ on $C_{i-2}$. We consider two cases.

Case 1. $|R(b_3)| = 4$.

Let $c_2$ be the out-vertex on $C_{i-2}$ such that $L(c_2) = R(c)$. See Figure 23(a). Since $|R(b_3)| = 4$, $C_{i-2}(c, c_2) = \emptyset$. Let $c'_1, c'_2$ denote the in-vertices on $C_{i-2}$ such that $R(c'_1) = B(c) = L(c'_2)$, and let $d_1, d_2$ be the out-vertices on $C_{i-3}$ such that $R(d_1) = B(c) = L(d_2)$. Since $i - 2 \geq 22$ and by Lemmas 6.1 and 6.2, $|C_{i-3}(d_1, d_2)| \geq 2$. Hence $|B(c)| = |B(c_2)| \geq 8$. By Lemma 5.4, $C_{i-2}(c'_1, c) = \emptyset = C_{i-2}(c_2, c'_2)$. So $|B(b_1)| = |A(c'_1)| \geq 6$ and $|R(c'_1)| = |B(c)| \geq 8$. Hence $|L(c'_1)| = 4$, as otherwise $\ell(c'_1) \geq (5, 6, 8)$ and $c'_1$ would be non-positive.

![Figure 23](image-url)
Therefore, \( c_1' \) is adjacent to \( d_1 \). So \( |L(d_1)| = |L(c_1')| = 4 \). Note that \( |R(d_1)| = |R(c_1')| \geq 8 \). Since \( i - 3 \geq 21 \) and since \( L(d_1) \cap C_{i-3} \) consists of two consecutive out-vertices on \( C_{i-3} \), it follows from Lemmas 6.1 and 6.2 that \( |B(d_1)| \geq 8 \). Thus \( \ell(d_1) \geq (4, 8, 8) \) and \( d_1 \) is non-positive, a contradiction.

**Case 2.** \( |R(b_3)| \geq 5 \).

Then \( |L(b_3)| = |R(b_3)| = 5 \), for otherwise, \( \ell(b_3) \geq (5, 6, 8) \) and \( b_3 \) would be non-positive. Also \( |R(b_1)| \leq 9 \), or else, \( \ell(b_3) \geq (5, 5, 10) \) and \( b_3 \) would be non-positive.

Let \( b_1' \) be the in-vertex on \( C_{i-1} \) such that \( R(b_1') = B(b_1) \), and let \( c_1, c \) be the out-vertices on \( C_{i-2} \) such that \( R(c_1) = L(c) = B(b_1) \). See Figure 23(b). Since \( |L(b_3)| = 5 \), \( C_{i-2}(c_1, c) = \emptyset \), \( b_3 \) is adjacent to \( c \), and \( b_1' \) is adjacent to \( c_1 \).

Let \( c_1', c' \) be the in-vertices on \( C_{i-2} \) such that \( R(c_1') = L(c') = B(c_1) \). Let \( d_1, d_2 \) denote the out-vertices on \( C_{i-3} \) such that \( R(d_1) = L(d_2) = B(c_1) \). Since \( i - 2 \geq 22 \) and by Lemma 5.4, \( C_{i-2}(c_1', c_1) = \emptyset = C_{i-2}(c, c') \). Since \( i - 3 \geq 21 \) and by Lemmas 6.1 and 6.2, \( |C_{i-3}(d_1, d_2)| \geq 2 \). Hence \( |B(c_1)| = |B(c)| = |L(c')| \geq 8 \).

Since \( |R(b_3)| = 5 \), \( b_3 \) is adjacent to \( c \), and \( C_{i-2}(c, c') = \emptyset \), we have \( |R(c) \cap C_{i-2}| = 3 \). Thus \( c' \) is adjacent to \( d_2 \), for otherwise, \( \ell(c') \geq (5, 6, 8) \) and \( c' \) would be non-positive. So \( |R(d_2)| \geq 5 \). Further, if \( |R(d_2)| = 5 \), then \( |B(d_2)| \geq 6 \). Therefore, \( \ell(d_2) \geq (5, 6, 8) \) and \( d_2 \) is non-positive, a contradiction.

**Lemma 6.4.** Let \( i \geq 24 \), and let \( a_1 \) and \( a_2 \) be consecutive in-vertices on \( C_i \) such that \( R(a_1) = L(a_2) \). Then \( |C_i(a_1, a_2)| \leq 1 \).

**Proof.** Suppose \( |C_i(a_1, a_2)| \geq 2 \). Then by Lemma 5.4, \( |C_i(a_1, a_2)| = 2 \). Let \( b_1, b_2 \) denote the out-vertices on \( C_{i-1} \) such that \( R(b_1) = L(b_2) = R(a_1) \). See Figure 24. By Lemmas 6.1 and 6.2, \( |C_{i-1}(b_1, b_2)| \geq 2 \), and so, \( |R(b_1)| = |L(b_2)| \geq 8 \). Let \( b_3, b_4 \) be the vertices in \( C_{i-1}(b_1, b_2) \) such that \( C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2) \). By Lemma 6.3, both \( b_3 \) and \( b_4 \) are out-vertices. Let \( c_1, c \) be the out-vertices on \( C_{i-2} \) such that \( R(c_1) = B(b_1) = L(c) \). Since \( i - 1 \geq 23 \), it follows from Lemmas 6.1 and 6.2 that \( |C_{i-2}(c_1, c)| \geq 2 \). Hence \( |B(b_1)| = |B(b_3)| \geq 8 \).

By Lemma 5.4, \( C_{i-1}(b_1', b_1) = \emptyset \). Thus \( |L(b_1)| \geq 4 \). Therefore \( \ell(b_1) \geq (4, 8, 8) \) and \( b_1 \) is non-positive, a contradiction.

**FIGURE 24.** Proof of Lemma 6.4.
7. PROOF OF THE MAIN RESULT

In this section, we complete the proof of Theorem 1.1. Let $G$ be a positively curved, cubic, infinite plane graph. By Theorem 2.1, $G$ has a nice sequence $(C_0, C_1, \ldots)$. By Theorem 2.2, we may assume that $G$ is nicely embedded with respect to $(C_0, C_1, \ldots)$.

**Lemma 7.1.** For $i \geq 26$, $|C_{i-1}| > |C_i|$.

**Proof.** Let $a_1, a_2, \ldots, a_n$ denote the in-vertices occurring on $C_i$ in that clockwise order. For each $j \in \{1, \ldots, n\}$, let $b_j, b'_j$ be the out-vertices on $C_{i-1}$ such that $R(b_j) = R(a_j)$ and $L(b'_j) = L(a_j)$. See Figure 25(a). For convenience, let $b_{n+1} = b_1$, $b'_{n+1} = b'_1$, and $a_{n+1} = a_1$.

To prove the lemma, it suffices to show that, for each $j \in \{1, \ldots, n\}$, $|C_{i-1}(b_j, b'_j)| > |C_i(a_j, a_{j+1})|$, and there is some $k \in \{1, \ldots, n\}$ such that $|C_{i-1}(b_k, b'_k)| > |C_i(a_k, a_{k+1})|$.

By Lemma 6.4, $|C_i(a_j, a_{j+1})| \leq 1$.

If $|C_i(a_j, a_{j+1})| = 0$, then clearly $|C_{i-1}(b_j, b'_j)| \geq |C_i(a_j, a_{j+1})|$. Now assume that $|C_i(a_j, a_{j+1})| = 1$. Since $i - 1 \geq 25$, it follows from Lemma 6.4 that $C_{i-1}$ has no consecutive out-vertices. Hence $|C_{i-1}(b_j, b'_j)| \geq 1$. So $|C_{i-1}(b_j, b'_j)| \geq |C_i(a_j, a_{j+1})|$.

Hence we may assume that $b_j = b'_j$ for all $j \in \{1, \ldots, n\}$, for otherwise, we have $|C_{i-1}| > |C_i|$. Because $G$ is connected and $(C_0, C_1, \ldots)$ is an infinite sequence, there is some $k \in \{1, \ldots, n\}$ such that $|C_i(a_k, a_{k+1})| = 1$. So $|R(b_k)| \geq 6$. See Figure 25(b). Note that $|B(b_k)| \geq 5$.

If $|B(b_k)| = 5$, then $|B(b_k) \cap C_{i-2}| = 2$ and $C_{i-2}$ has consecutive out-vertices, contradicting Lemma 6.4 (because $i - 2 \geq 24$). So $|B(b_k)| \geq 6$. Hence $|L(b_k)| \leq 5$, or else $\ell(b_k) \geq (6, 6, 6)$ and $b_k$ would be non-positive. In fact, $|L(b_k) \cap C_{i-1}| \geq 3$ by Lemma 6.4 (since $i - 1 \geq 25$ and $C_{i-1}$ has no consecutive out-vertices). So $|C_{i-1}(b_{k-1}, b'_k)| = |C_{i-1}(b_{k-1}, b_k)| > |C_i(a_{k-1}, a_k)|$. Therefore, $|C_{i-1}| > |C_i|$.

It is now easy to see that Theorem 1.1 holds because of the contradiction caused by Lemma 7.1 and the infinite sequence $(C_0, C_1, \ldots)$.

![Figure 25. Proof of Lemma 7.1.](image-url)
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REFERENCES


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