Integral complete $r$-partite graphs

Ligong Wang\textsuperscript{1}, Xueliang Li\textsuperscript{2} and C. Hoede\textsuperscript{3}

\textsuperscript{1}Department of Mathematics and Information Sciences and Department of Computer Science and Engineering, Northwestern Polytechnical University, Xi’an, Shaanxi 710072, People’s Republic of China. E-mail: ligongwangnpu@yahoo.com.cn
\textsuperscript{2}Center for Combinatorics, Nankai University, Tianjin, 300071, People’s Republic of China. E-mail: x.li@eyou.com
\textsuperscript{3}Faculty of Mathematical Science, University of Twente, P.O.Box 217, 7500 AE Enschede, The Netherlands. E-mail: hoede@math.utwente.nl

Abstract

A graph is called integral if all the eigenvalues of its adjacency matrix are integers. In this paper, we give a useful sufficient and necessary condition for complete $r$-partite graphs to be integral, from which we can construct infinite many new classes of such integral graphs. It is proved that the problem of finding such integral graphs is equivalent to the problem of solving some Diophantine equations. The discovery of these integral complete $r$-partite graphs is a new contribution to the search of such integral graphs. Finally, we propose several basic open problems for further study.

Key Words: Integral graph, Complete $r$-partite graph, Diophantine equation, Graph spectrum.

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1 Introduction

We shall consider only simple undirected graphs (i.e. undirected graphs without loops or multiple edges). For a graph $G$, let $V(G)$ denote the vertex set and $E(G)$ the edge set.

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The characteristic polynomial \(|xI - A|\) of the adjacency matrix \(A\) (or \(A(G)\)) of \(G\) is called the characteristic polynomial of \(G\) and denoted by \(P(G, x)\). The spectrum of \(A(G)\) is also called the spectrum of \(G\).

The notion of integral graphs was first introduced by F. Harary and A.J. Schwenk in 1974 (see [9]). A graph \(G\) is called integral if all the zeros of the characteristic polynomial \(P(G, x)\) are integers. In general, the problem of characterizing integral graphs seems to be difficult. Thus it makes sense to restrict our investigations to some interesting families of graphs, for instance, cubic graphs [3, 19], complete tripartite graphs [17], graphs with three eigenvalues [16], graphs with maximum degree 4 [1], etc. Trees present another important family of graphs for which the problem has been considered in [4, 5, 6, 10, 11, 12, 13, 14, 15, 20, 21, 23, 24, 25]. Other results on integral graphs can be found in [7, 8, 22]. For all other facts on graph spectra (or terminology), see [7, 8].

A complete \(r\)-partite graph \(K_{p_1,p_2,\ldots,p_r}\) is a graph with a set \(V = V_1 \cup V_2 \cup \cdots \cup V_r\) of \(p_1 + p_2 + \cdots + p_r (= n)\) vertices, where \(V_i\)'s are nonempty disjoint sets, \(|V_i| = p_i\) for \(1 \leq i \leq r\), such that two vertices in \(V\) are adjacent if and only if they belong to different \(V_i\)'s. An infinite family of integral complete tripartite graphs was constructed in [17], where the author mentioned the general problem on integral complete multipartite graphs. He thought that it is possible that for \(r > 3\) there also exist an infinite number of integral complete \(r\)-partite graphs. But he did not find such integral graphs. The authors of [1] thought that the general problem seems to be intractable. In this paper, we give a useful sufficient and necessary condition for complete \(r\)-partite graphs to be integral, from which we can construct infinite many new classes of such integral graphs. It is proved that the problem of finding such integral graphs is equivalent to the problem of solving some Diophantine equations. The discovery of these integral complete \(r\)-partite graphs is a new contribution to the search of such integral graphs. In fact, M. Roitman’s result on the integral complete tripartite graphs is generalized in this paper (see also M. Roitman, An infinite family of integral graphs, Discrete Math. 52(2-3)(1984), 313-315. MR 86a:05089). Finally, we propose several basic open problems for further study.

2 A sufficient and necessary condition for complete \(r\)-partite graphs to be integral

In this section, we shall give a useful sufficient and necessary condition for complete \(r\)-partite graphs to be integral.

\[ P(K_{p_1,p_2,\ldots,p_r}, x) = x^{n-r}(1 - \sum_{i=1}^{r} \frac{p_i}{x + p_i}) \prod_{j=1}^{r} (x + p_j). \]

Assume that the number of distinct integers of \(p_1, p_2, \cdots, p_r\) is \(s\). Without loss of generality, assume that the first \(s\) ones are the distinct integers such that \(p_1 < p_2 < \cdots < p_s\). Suppose that \(a_i\) is the multiplicity of \(p_i\) for each \(i = 1, 2, \cdots, s\). The complete \(r\)-partite
graph $K_{p_1,p_2,\ldots,p_s}=K_{p_1,\ldots,p_s}$ is also denoted by $K_{a_1,p_1,a_2,p_2,\ldots,a_s,p_s}$, where $r = \sum_{i=1}^s a_i$ and $|V| = n = \sum_{i=1}^s a_ip_i$.

**Example 2.2.** (see [8]) For the complete r-partite graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1,p_1,a_2,p_2,\ldots,a_s}$ on $n$ vertices, if $s = 2$, $a_1 = a_2 = 1$, then $K_{p_1,p_2}$ is integral if and only if $p_1p_2$ is a perfect square.

**Corollary 2.3.** For the complete r-partite graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1,p_1,a_2,p_2,\ldots,a_s}$ on $n$ vertices, we have

$$P(K_{a_1,p_1,a_2,p_2,\ldots,a_s},x) = x^{n-r} \prod_{i=1}^{s} (x + p_i)^{a_i-1} \prod_{i=1}^{s} (x + p_i) - \sum_{j=1}^{s} a_jp_j \prod_{i=1,i\neq j}^{s} (x + p_i).$$

The following Theorem 2.4 is immediate.

**Theorem 2.4.** The complete r-partite graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1,p_1,a_2,p_2,\ldots,a_s}$ on $n$ vertices is integral if and only if

$$\prod_{i=1}^{s} (x + p_i) - \sum_{j=1}^{s} a_jp_j \prod_{i=1,i\neq j}^{s} (x + p_i) = 0. \tag{1}$$

has no other roots but integral ones.

We shall discuss Eqn.(1) to get more information. First, we divide both sides of Eqn.(1) by $\prod_{i=1}^{s} (x + p_i)$, and obtain

$$\sum_{i=1}^{s} \frac{a_ip_i}{x + p_i} = 1. \tag{2}$$

Let $F(x) = \sum_{i=1}^{s} \frac{a_ip_i}{x + p_i} - 1$. Clearly, $-p_i$ are not roots of Eqn.(1) for $1 \leq i \leq s$. Hence, all the solutions of Eqn.(1) are the same as those of Eqn.(2). Now we consider the roots of $F(x)$ over the set of real numbers. Note that each $p_i$ is discontinuous point of $F(x)$. For $1 \leq i \leq s$, we have that $F(-p_i - 0) = -\infty$, $F(-p_i + 0) = +\infty$, $F(-\infty) = F(+\infty) = -1$, $F'(x) = -\sum_{i=1}^{s} \frac{a_ip_i}{(x + p_i)^2}$. We deduce that $F(x)$ is strictly monotone decreasing on each of the continuous intervals over the set of real numbers. Using Zero Point Theorem of Mathematical Analysis, we get that all the roots of $F(x)$ are real and it has $s$ distinct roots. Let all the roots of $F(x)$ are given by $-\infty < u_s < u_{s-1} < \cdots < u_1 < +\infty$, then we obtain

$$-p_s < u_s < -p_{s-1} < u_{s-1} < \cdots < -p_2 < u_2 < -p_1 < u_1 < +\infty. \tag{3}$$

On the other hand, we note that Eqn.(2) can be written as

$$\frac{a_1p_1}{x + p_1} + \frac{a_2p_2}{x + p_2} + \cdots + \frac{a_sp_s}{x + p_s} = 1. \tag{4}$$

From the above discussion, we have
Theorem 2.5. The complete $r$-partite graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1,a_2,a_3,\ldots,a_n}$ on $n$ vertices is integral if and only if all the solutions of Eqn. (4) are integers. Moreover, there exist integers $u_1, u_2, \ldots, u_s$ satisfying Ineqn. (3) such that the following linear equation system on $a_1, a_2, \ldots, a_s$

$$
\begin{align*}
\frac{a_1 p_1}{u_1 + p_1} + \frac{a_2 p_2}{u_1 + p_2} + \cdots + \frac{a_s p_s}{u_1 + p_s} &= 1 \\
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
\frac{a_1 p_1}{u_s + p_1} + \frac{a_2 p_2}{u_s + p_2} + \cdots + \frac{a_s p_s}{u_s + p_s} &= 1
\end{align*}
$$

has positive integral solutions $(a_1, a_2, \ldots, a_s)$.

Theorem 2.6. The complete $r$-partite graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1,a_2,a_3,\ldots,a_n}$ on $n$ vertices is integral if and only if there exist integers $u_i$ and positive integers $p_i$ $(i = 1, 2, \ldots, s)$ such that $-p_s < u_s < -p_{s-1} < u_{s-1} < \cdots < -p_2 < u_2 < -p_1 < u_1 < +\infty$ and

$$a_k = \frac{\prod_{i=1}^{s} (p_k + u_i)}{p_k \prod_{i=1,i \neq k}^{s} (p_k - p_i)}$$

are positive integers.

Proof. From Cauchy’s result on determinants in [2], we know that

$$\begin{vmatrix}
\frac{1}{\alpha_1 + \beta_1} & \cdots & \frac{1}{\alpha_1 + \beta_s} \\
\vdots & \ddots & \vdots \\
\frac{1}{\alpha_s + \beta_1} & \cdots & \frac{1}{\alpha_s + \beta_s}
\end{vmatrix} = \frac{\prod_{1 \leq i < j \leq s} (\alpha_j - \alpha_i)(\beta_j - \beta_i)}{\prod_{1 \leq i < j \leq s} (\alpha_i + \beta_j)}$$

The determinant of the coefficient matrix $D$ of the linear equation system (5) is the following:

$$|D| = \prod_{i=1}^{s} p_i \prod_{1 \leq i < j \leq s} (u_j - u_i)(p_j - p_i)$$

$$\frac{\prod_{1 \leq i < j \leq s} (u_i + p_j)}{\prod_{1 \leq i < j \leq s} (u_i + p_j)} \neq 0.$$

$$|D_k| = \prod_{i=1}^{s} p_i \prod_{1 \leq i < j \leq s} (u_j - u_i)(p_j - p_i)$$

$$\lim_{p_k \to +\infty} |D| = \prod_{i=1,i \neq k}^{s} (u_i + p_j) \prod_{1 \leq i < j \leq s}^{s} (p_k - p_i)$$
for \( k = 1, 2, \cdots, s \).

By using the well-known Cramer’s Rule to solve the linear equation system (5) on \( a_1, a_2, \cdots, a_s \), we obtain that

\[
a_k = \frac{\prod_{i=1}^{s}(p_k + u_i)}{p_k \prod_{i=1, i \neq k}^{s}(p_k - p_i)}, \quad (k = 1, 2, \cdots, s)
\]

Because \(-p_s < u_s < -p_{s-1} < u_{s-1} < \cdots < -p_2 < u_2 < -p_1 < u_1 < +\infty\) and \( p_i \geq 1 \) for \( i = 1, 2, \cdots, s \), we can deduce that \( a_k > 0 \) (\( k = 1, 2, \cdots, s \)) and \( u_1 > 0 \).

The remaining part of the theorem can be easily proved from Lemma 2.1 and Theorems 2.4 and 2.5.

\[\square\]

**Corollary 2.7.** If the complete \( r \)-partite graph \( K_{p_1,p_2,\cdots,p_r} = K_{a_1,a_2,a_3,\cdots,a_s,p_s} \) on \( n \) vertices is integral, then we have the following results.

1. \( a_k > 0 \) for \( k = 1, 2, \cdots, s \), and \( u_1 > 0 \).
2. \( \sum_{i=1}^{s} u_i = \sum_{i=1}^{s} (a_i - 1)p_i \).
3. \( \prod_{i=1}^{s} u_i = \prod_{i=1}^{s} p_i (1 - \sum_{i=1}^{s} a_i) \).
4. \( Spec(K_{a_1,a_2,a_3,\cdots,a_s,p_s}) = \)

\[
\left( \begin{array}{cccccc}
-p_s & u_s & -p_{s-1} & u_{s-1} & \cdots & -p_2 & u_2 & -p_1 \\
 a_s - 1 & 1 & a_{s-1} - 1 & 1 & \cdots & a_2 - 1 & 1 & a_1 - 1 \sum_{i=1}^{s} a_i (p_i - 1) & 1
\end{array} \right).
\]

**Proof.** It is easy to check the correctness from Corollary 2.3. \[\square\]

The following lemma is due to a referee.

**Lemma 2.8.** Denote

\[
F_{\tilde{a},\tilde{p}}(x) := \sum_{i=1}^{s} \frac{a_i p_i}{x + p_i},
\]

\[
\Phi_{\tilde{a},\tilde{p}}(x) := \left( \prod_{i=1}^{s} (x + p_i) \right) \left( 1 - F_{\tilde{a},\tilde{p}}(x) \right).
\]

where vectors

\[
\tilde{a} := (a_1, a_2, \cdots, a_s), \quad \tilde{p} := (p_1, p_2, \cdots, p_s) \in \mathbb{Z}^s.
\]

Let \( q \) be a non-zero integer. Then \( u \) is an integral root of \( \Phi_{\tilde{a},\tilde{p}}(x) \) if and only if \( u/q \) is an integral root of \( \Phi_{\tilde{a},\tilde{p}}(x) \).

**Proof.** It is easy to see that \( v \) is a root of \( \Phi_{\tilde{a},\tilde{p}}(x) \) if and only if \( qv \) is a root of \( \Phi_{\tilde{a},\tilde{p}}(x) \). Therefore if all the roots of \( \Phi_{\tilde{a},\tilde{p}}(x) \) are integers, then the roots of \( \Phi_{\tilde{a},\tilde{p}}(x) \) are integers as well.

Assume now that all roots of \( \Phi_{\tilde{a},\tilde{p}}(x) \) are integral and let \( v \) be one of them. Then \( v/q \) is a rational root of \( \Phi_{\tilde{a},\tilde{p}}(x) \). Since \( \Phi_{\tilde{a},\tilde{p}}(x) \) is a monomial polynomial with integral coefficients, its integral roots should be integers. Therefore \( v/q \in \mathbb{Z} \). \[\square\]

From the above lemma we can obtain the following result.
Theorem 2.9. For any positive integer \( q \), the complete \( r \)-partite graph \( K_{a_1 p_1 q, a_2 p_2 q, \ldots, a_s p_s q} \) is integral if and only if the complete \( r \)-partite graph \( K_{p_1, p_2, \ldots, p_r} = K_{a_1 p_1, a_2 p_2, \ldots, a_s p_s} \) is integral.

Remark 2.10. The above Theorem 2.9 shows that it is reasonable to study Eqn.(2) only when \((p_1, p_2, \cdots, p_s) = 1\). Let’s call such a vector primitive. So, in general, the primitive vectors are the only ones which are of interest.

3 Integral complete \( r \)-partite graphs

In this section, we shall construct infinite many new classes of integral complete \( r \)-partite graphs \( K_{p_1, p_2, \cdots, p_r} = K_{a_1 p_1, a_2 p_2, \cdots, a_s p_s} \) from Theorem 2.5 or 2.6. They are different from those of [7, 8, 17].

The idea of constructing such integral graph is as follows: First, we properly choose positive integers \( p_1, p_2, \cdots, p_s \). Then, we try to find integers \( u_i \) \((i = 1, 2, \cdots, s)\) satisfying Ineqn.(3) such that there are positive integral solutions \((a_1, a_2, \cdots, a_s)\) for the linear equation system (5) (or such that all \( a_k \)’s of (6) are positive integers). Finally, we obtain positive integers \( a_1, a_2, \cdots, a_s \) such that all the solutions of Eqn.(4) are integers. Thus, we have constructed many new classes of integral graphs \( K_{a_1 p_1, a_2 p_2, \cdots, a_s p_s} \).

Example 3.1. Let \( p_1 = 1, p_2 = 9 \) and \( u_2 = -4 \). If \( u_1 = 72t - 9 \) and \( t \) is a positive integer, then \( K_{p_1, p_2, \cdots, p_r} = K_{a_1 p_1, a_2 p_2} \) is integral for infinite many positive integers \( a_1, a_2 \) given by (7) and (8).

Proof. From Theorem 2.6, we have that

\[
a_1 = \frac{(p_1 + u_1)(p_1 + u_2)}{p_1(p_1 - p_2)} = \frac{3}{8}(u_1 + 1) \tag{7}
\]

and

\[
a_2 = \frac{(p_2 + u_1)(p_2 + u_2)}{p_2(p_2 - p_1)} = \frac{5}{72}(u_1 + 9) \tag{8}
\]

So, \( K_{a_1 p_1, a_2 p_2} \) is integral if and only if \( a_1 \) and \( a_2 \) are positive integers. From (7) and (8), we get the Diophantine equation

\[
27a_2 - 5a_1 = 15. \tag{9}
\]

From elementary number theory knowledge, all the positive integral solutions of Eqn.(9) are given by \( a_1 = 27t - 3, a_2 = 5t \), and \( u_1 = 72t - 9 \), where \( t \) is a positive integer.

Hence, \( K_{p_1, p_2, p_3, \cdots, p_r} = K_{a_1 p_1, a_2 p_2} \) is integral for the above infinite many integers \( a_1 \) and \( a_2 \).

The following Lemma 3.2 can be found in [18].

Lemma 3.2. Let \( a, b \) and \( c \) be integers with \( d = (a, b) \), we have
(1) If $d \nmid c$, then the linear Diophantine equation in two variables

$$ax + by = c$$

(10)

does not have integral solutions.

(2) If $d \mid c$, then there are infinite many integral solutions for Eqn.(10). Moreover, if $x = x_0, y = y_0$ is a particular solution of Eqn.(10), then all its solutions are given by

$$x = x_0 + (b/d)t, \quad y = y_0 - (a/d)t$$

where $t$ is an integer.

**Theorem 3.3.** For $s = 2$, let $p_1 < p_2$. Then $K_{a_1,p_1,a_2,p_2}$ is integral if and only if one of the following two conditions holds:

(1) When $(m, k) = 1$, let $p_1 = m, p_2 = m + k, m \geq 1, k \geq 2, 1 \leq q < k$, where $m, k$ and $q$ are positive integers. Then, $a_1$ and $a_2$ must be positive integral solutions for the Diophantine equation

$$q(m + k)a_2 - m(k - q)a_1 = q(k - q).$$

(11)

(2) When $(m, k) = d \geq 2$, let $p_1 = m, p_2 = m + k, m = m_1 d, k = k_1 d, (m_1, k_1) = 1, q = q_1 d, 1 \leq q_1 < k_1, m_1, k_1$, where $q_1$ and $d$ are positive integers. Then, $a_1$ and $a_2$ must be positive integral solutions for the Diophantine equation

$$q_1(m_1 + k_1)a_2 - m_1(k_1 - q_1)a_1 = q_1(k_1 - q_1).$$

(12)

**Proof.** Because $p_1 < p_2$, from Theorem 2.6, we know $K_{a_1,p_1,a_2,p_2}$ is integral if and only if there exist integers $u_1, u_2$ and positive integers $p_1, p_2$ satisfying $-p_2 < u_2 < -p_1 < u_1 < +\infty$ such that $a_1$ and $a_2$ are positive integers, where $a_k = \frac{\prod_{i=1}^{k}(p_k + u_i)}{\prod_{i=1, i \neq k}(p_k - p_i)}$ for $k = 1, 2$.

Hence, we choose $p_1 = m, p_2 = m + k, u_2 = -(m + q), m \geq 1, k \geq 2, 1 \leq q < k$, where $m, k$ and $q$ are positive integers, and we have

$$a_1 = \frac{q(m + u_1)}{mk}, \quad a_2 = \frac{(m + k + u_1)(k - q)}{k(m + k)}.$$

Hence, we get Eqn.(11). From Lemma 3.2, we know there are solutions for Eqn.(11) if and only if $d_1 \mid q(k - q)$, where $d_1 = (q(m + k), m(k - q))$.

Now, we discuss the following two cases.

**Case 1.** When $(m, k) = 1$, we have $(m + k, m) = 1$, and $d_1 \mid q(k - q)$. Moreover, there are solutions for Eqn.(11). From Lemma 3.2 and the condition $(m, k) = 1$, we know that there are infinite many integral solutions for Eqn.(11). Therefore, there are infinite many positive integral solutions $(a_1, a_2)$ for Eqn.(11).

**Case 2.** When $(m, k) = d \geq 2$, let $m = m_1 d, k = k_1 d, (m_1, k_1) = 1$, where $m_1, k_1$ and $d$ are positive integers. We have $(m_1 + k_1, m_1) = 1, d_1 = (qd(m_1 + k_1), m_1 d(k_1 d - q))$. 


If \( d_1|q(k-q) = q(k_1d-q) \), then \( d|q \). Thus, let \( q = q_1d \), \( 1 \leq q_1 < k_1 \), where \( q_1 \) is a positive integer. We can reduce Eqn.(11) into Eqn.(12). Hence, from Lemma 3.2 and the condition \((m_1,k_1)=1\), we know that there are infinite many integral solutions for Eqn.(12). Therefore, there are infinite many positive integral solutions \((a_1,a_2)\) for Eqn.(12).

Thus, the theorem is proved. \(\blacksquare\)

**Example 3.4.** (1) For \( s = 3 \), let \( p_1 = 1, p_2 = 5, p_3 = 9 \), \( u_2 = -3 \) and \( u_3 = -7 \). If \( u_1 = 120t-105 \), \( t \) is a positive integer, then \( K_{p_1,p_2,\ldots,p_r} = K_{a_1,p_1,a_2,p_2,a_3,p_3} \) is integral for infinite many positive integers \( a_1,a_2 \) and \( a_3 \).

(2) For any positive integer \( q \), if \( s = 3 \), let \( p_i' = p_iq \) and \( u_i' = u_ig \) for \( i = 1,2,3 \), where \( p_i, u_i \) and \( a_i \) \( (i = 1,2,3) \) are the same as those of (1) in Example 3.4, then \( K_{a_1,p_1',a_2,p_2,a_3,p_3'} = K_{a_1,p_1,a_2,p_2,a_3,p_3} \) is integral, too.

**Proof.** (1). From Theorem 2.6, we have that

\[
a_1 = \frac{3}{8}(u_1 + 1), \tag{13}
\]
\[
a_2 = \frac{1}{20}(u_1 + 5), \tag{14}
\]
\[
a_3 = \frac{1}{24}(u_1 + 9). \tag{15}
\]

So, \( K_{a_1,p_1,a_2,p_2,a_3,p_3} \) is integral if and only if \( a_1, a_2 \) and \( a_3 \) are positive integers. By (14) and (15), we get the Diophantine equation

\[
6a_3 - 5a_2 = 1. \tag{16}
\]

From elementary number theory knowledge, all the positive integral solutions of Eqn.(16) are given by \( a_2 = 6t - 5, a_3 = 5t - 4 \), where \( t \) is a positive integer, from (13) and (14), we have \( u_1 = 120t - 105, a_1 = 45t - 39 \), where \( t \) is a positive integer.

Hence, when \( p_1 = 1, p_2 = 5, p_3 = 9, a_1 = 45t - 39, a_2 = 6t - 5, a_3 = 5t - 4 \), where \( t \) is a positive integer, \( K_{p_1,p_2,\ldots,p_r} = K_{a_1,p_1,a_2,p_2,a_3,p_3} \) is integral.

(2). From Theorem 2.9 and (1) of Example 3.4, it is easy to prove \( K_{p_1',p_2',\ldots,p_r'} = K_{a_1,p_1',a_2,p_2,a_3,p_3'} \) is integral, too. \(\blacksquare\)

**Example 3.5.** (1) (see [17]) For \( s = 3 \), let \( p_1 = 4u^2(v^2 + u^2)^3, p_2 = 3u^2v^2(u^2 + 6uv + v^2)(-u^2 + 6uv - v^2), p_3 = 4v^2(u^2 + v^2)^3 \) such that \( (3 - \sqrt{8})v < u < v \), and let \( u_1 = 24u^2v^2(u^2 + v^2)^2, u_2 = -2uv(u^2 + v^2)^2(-u^2 + 6uv - v^2), u_3 = -2uv(u^2 + v^2)^2(u^2 + 6uv + v^2) \) such that \( u,v \) are positive integers. Then \( K_{p_1,p_2,\ldots,p_r} = K_{a_1,p_1,a_2,p_2,a_3,p_3} \) is integral for \( a_1 = a_2 = a_3 = 1 \).

(2) For any positive integer \( q \), if \( s = 3 \), let \( p_1, p_2 \) and \( p_3 \) be the same as those of (1) in Example 3.5, then \( K_{a_1,p_1,a_2,p_2,a_3,p_3} = K_{p_1,p_2,a_3} \) is integral, too.
Proof. (1). The condition $0 < (3 - \sqrt{38})v < u < v$, ensures $-u^2 + 6uv - v^2 > 0$.

$$p_3 - p_2 = v^2(7u^2 + v^2)(-u^2 - 3uw + 2v^2)(-u^2 + 3uv + 2v^2) = v^2(7u^2 + v^2)(u + v)(v - u) + 2v^3[u(3v - u) + 2v^2] > 0,$$

$$p_2 - p_1 = -u^2(u^2 + 7v^2)(2u^2 - 3uw - v^2)(2u^2 + 3uv - v^2) = -u^2(u^2 + 7v^2)(u + v)(u - v) - 2uv(2u^2 + 3uv - v^2) > 0,$$

$$p_3 - p_1 = -4(u - v)(u + v)(u^2 + v^2)^3 > 0,$$

$$p_1 + u_2 = 2u(u - v)(2u^2 + 3uv - v^2)(u^2 + v^2)^2 < 0,$$

$$u_2 + p_2 = uv(u^2 - 3uv - 2v^2)(2u^2 - 3uv - v^2)(u^2 - 6uv + v^2) > 0,$$

$$p_2 - u_3 = uv(u^2 + 3uv - 2v^2)(2u^2 - 3uv - v^2)(u^2 + 6uv + v^2) > 0,$$

$$u_3 + p_3 = 2u(u + v)(u^2 + v^2)^2(-u^2 - 3uv + 2v^2) > 0,$$

$$p_1 + u_1 = 4u^2(u^2 + v^2)^2(u^2 + 7v^2),$$

$$p_1 + u_2 = 2u(u - v)(2u^2 + 3uv - v^2)(u^2 + v^2)^2,$$

$$p_1 + u_3 = 2u(u + v)(2u^2 - 3uv - v^2)(u^2 + v^2)^2,$$

$$p_2 + u_1 = 3u^2(7u^2 + v^2)(u^2 + 7v^2),$$

$$p_2 + u_2 = uv(u^2 - 3uv - 2v^2)(2u^2 + 3uv - v^2)(u^2 - 6uv + v^2),$$

$$p_2 + u_3 = -uv(u^2 + 3uv - 2v^2)(2u^2 - 3uv - v^2)(u^2 + 6uv + v^2),$$

$$p_3 + u_1 = 4u^2(u^2 + v^2)^2(7u^2 + v^2),$$

$$p_3 + u_2 = 2u(u - v)(u^2 - 3uv - 2v^2)(u^2 + v^2)^2,$$

$$p_3 + u_3 = 2u(u + v)(u^2 - 3uv - 2v^2)(u^2 + v^2)^2,$$

Hence, we get that $-p_3 < u_3 < -p_2 < u_2 < -p_1 < u_1 < +\infty$, and

$$a_k = \frac{\prod_{i=1}^{3}(p_k + u_i)}{p_k \prod_{i=1,i\neq k}^{3}(p_k - p_i)} = 1, (k = 1, 2, 3).$$

From Theorem 2.6, we know that $K_{a_1, a_2, p_2, a_3, p_3} = K_{p_1, p_2, p_3}$ is integral.

(2). From Theorem 2.9 and (1) of Example 3.5, it is easy to prove that $K_{a_1, p_2, a_2, p_2, q}$, $a_3, p_3 = K_{p_1, p_2, p_3}$ is integral, too.

Theorem 3.6. For $s = 3$, let $q$ be any positive integer, and let $p_i$ ($i = 1, 2, 3$) be positive integers in the following Table 1, $a_1 = a_2 = a_3 = 1$, and $u_i$ ($i = 1, 2, 3$) be those of Theorem 2.6. Then $K_{a_1, p_2, a_2, p_2, a_3, p_3} = K_{p_1, p_2, p_3, p_4}$ is integral.

Proof. It is easy to check the correctness by making use Theorems 2.4, 2.5 or 2.6 and 2.9.

Remark 3.7. An infinite family of integral complete tripartite graphs $K_{p_1, p_2, p_3}$ was constructed in [17]. In Table 1, by using a computer, we have found 34 solutions $(p_1, p_2, p_3)$, where $s = 3$ and $p_1 < p_2 < p_3$, $1 \leq p_1 \leq 50$, $p_1 + 1 \leq p_2 \leq p_1 + 50$, and $p_2 + 1 \leq p_3 \leq p_2 + 100$. We shall construct infinite many classes of such integral graphs from Theorems 2.4, 2.5, 2.6 and 2.9. They are different from those in existing literature (see [7, 8, 17]). We believe that it is useful for constructing other integral complete tripartite graphs. When $s = 3$, $p_1 < p_2 < p_3$ and $a_1 = a_2 = a_3 = 1$. For any positive integer $q$, the complete tripartite graph $K_{5q, 8q, 12q}$ is integral and

$$Spec(K_{5q, 8q, 12q}) = \begin{pmatrix}
-10q & -6q & 0 & 16q \\
1 & 1 & 25q - 3 & 1
\end{pmatrix}.$$
Table 1: Integral complete tripartite graph $K_{p_1,q,p_2,q,p_3,q}$, where $q$ is a positive integer.

<table>
<thead>
<tr>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$u_1$</th>
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<td>80</td>
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</tbody>
</table>

If $q = 1$, let $s = 3$, $p_1 < p_2 < p_3$ and $a_1 = a_2 = a_3 = 1$, we know that the complete tripartite graph $K_{5,8,12}$ is an integral one, the order of which is 25, which is much smaller than those given in [7, 8, 17].

**Remark 3.8.** Theorem 3.6 generates an infinite set of vectors $\vec{p} = (p_1, p_2, p_3)$ for which (2) has integral solutions only. But there is only finite number of primitive vectors in this infinite set (in general, the primitive vectors are the only ones which are of interest). The infinite series built in [17] gives an infinite series of the primitive solutions. Thus Theorem 3.6 is much weaker than the result of [17]. However, by analyzing Table 1 one can see that all its rows except the row $\vec{p} := (5, 13, 77)$ have the following property: $u_3/p_i = -2$ for a suitable $i \in \{1, 2, 3\}$. This observation gives a hint to a new infinite series of primitive triples $\vec{p}$, see the following, which is due to a referee.

Let $u_3 < u_2 < u_1$ be the roots of $F(x) = F_{\vec{p}}(x) = 0$. Set $v_3 := -u_3$, $v_2 := -u_2$, $v_1 := u_1$. Then $v_i$’s are positive integers which satisfy the following conditions:

$$v_1 = v_2 + v_3,$$
$$v_2^2 + v_2v_3 + v_3^2 = p_1p_2 + p_1p_3 + p_2p_3,$$
$$u_3/v_2v_3 = 2p_1p_2p_3.$$  \hfill (17)

Let’s look for solutions of (17) such that $v_3 = 2p_i$ for some $i=1,2,3$. Forgetting about
ordering of $p_i$'s we may assume that $v_3 = 2p_3$. Then (17) is equivalent to the following

$$p_1 + p_2 = 2v_3,$$
$$p_1p_2 = (v_2 + v_3)v_2.$$  \hspace{1cm} (18)

These equations have an integral solutions for $p_1, p_2$ if and only if $v_3^2 - (v_2 + v_3)v_2$ is a perfect square, say $m^2$. Then $p_1 = v_3 - m, p_2 = v_3 + m, p_3 = v_3/2$.

$$v_3^2 - (v_2 + v_3)v_2 = m^2 \iff$$
$$\iff x := \frac{v_3}{m}, \ y := \frac{v_2}{m}, \ x^2 - xy - y^2 = 1 \iff$$
$$\iff x = \frac{t^2 + 1}{t^2 + t - 1}, \ y = t(x - 1), \ t \in \mathbb{Q}.$$

We may assume that $m > 0$. It follows from $p_1 = m(x - 1) > 0$ and $v_2 > 0$ that $x > 1$ and $t > 0$. The first inequality is equivalent to

$$\frac{2 - t}{t^2 + t - 1} > 0 \iff \frac{\sqrt{5} - 1}{2} = 0.618.. < t < 2 \text{ or } t < -\frac{\sqrt{5} - 1}{2} = -1.618..$$

Thus $p_1 = v_3 - m = m(x - 1), p_2 = v_3 + m = m(x + 1), p_3 = v_3/2 = mx/2$ are non-negative.

Write $t = a/b$ where $(a, b) = 1$ and $a > 0, b > 0$ we obtain

$$\frac{v_3}{m} = x = \frac{a^2 + b^2}{a^2 + ab - b^2}; \ \frac{v_2}{m} = y = \frac{2ab - b^2}{a^2 + ab - b^2}.$$

After some routine transformations we obtain

$$p_1 = \frac{2b(2b - a)}{d}, \ p_2 = \frac{2a(2a + b)}{d}, \ p_3 = \frac{a^2 + b^2}{d};$$
$$u_1 = \frac{2b(b + 2a)}{d}, \ u_2 = -\frac{2a(2b - a)}{d}, \ u_3 = -\frac{2(a^2 + b^2)}{d};$$  \hspace{1cm} (19)

where $d := (2b(2b - a), 2a(2a + b), a^2 + b^2)$. Note that $d \in \{1, 2, 5, 10\}$.

Take for example $a = 2, b = 3$. Then $d = 1$ and

$$p_1 = 24, \ p_2 = 28, \ p_3 = 13, \ u_1 = 42, \ u_2 = -16, \ u_3 = -26.$$ 

This is one of the triple given Table 1. All triple given in this table except $(5, 12, 77)$ may be obtained from (19).

Note that the above numbering of $p_i$'s may not coincide with one fixed before.

Hence, we obtain the following result.

**Theorem 3.9.** For $s = 3$, let $q$ be any positive integer, and let $p_i$ and $u_i (i = 1, 2, 3)$ be positive integers in the above formulae (19), $a_1 = a_2 = a_3 = 1$. Then $K_{a_1, p_1q, a_2, p_2q, a_3, p_3q} = K_{p_1q, p_2q, p_3q}$ is integral.
Example 3.10. For any positive integer \( q \), if \( s = 3 \), let \( p_1 = q, p_2 = 3q, p_3 = 5q, u_2 = -2q, u_3 = -4q \), then there do not exist positive integers \( a_1, a_2, a_3 \) such that \( K_{p_1, p_2, \ldots, p_r} = K_{a_1, a_2, p_2, a_3, p_3} \) is integral.

Proof. When \( s = 3 \), \( p_1 = q, p_2 = 3q, p_3 = 5q, u_2 = -2q, u_3 = -4q \). Suppose that we can construct integral graphs \( K_{a_1, p_1, a_2, p_2, a_3, p_3} \). From Theorem 2.6, we know that \( K_{a_1, p_1, a_2, p_2, a_3, p_3} \) is integral if and only if there exist integers \( a_i \) and positive integers \( p_i \) \((i = 1, 2, 3)\) satisfying 

\[-p_3 < u_3 < -p_2 < u_2 < -p_1 < u_1 < +\infty \text{ such that } a_k = \frac{\prod_{i=1}^{3} (p_k + u_i)}{p_k \prod_{i=1, i \neq k} (p_k - p_i)} \quad (k = 1, 2, 3)\text{are positive integers.}

Hence, we obtain

\[
a_1 = \frac{3}{8q}(u_1 + q),
\]

\[
a_2 = \frac{1}{12q}(u_1 + 3q),
\]

\[
a_3 = \frac{3}{40q}(u_1 + 5q).
\]

By (21) and (22), we have

\[
20a_3 - 18a_2 = 3.
\]

From Lemma 3.2, we know that there are no integral solutions for Eqn.(23).

Hence, we can’t construct integral graphs \( K_{a_1, p_1, a_2, p_2, a_3, p_3} \).

Theorem 3.11. For the complete \( r \)-partite graph \( K_{p_1, p_2, \ldots, p_r} = K_{a_1, p_1, a_2, p_2, \ldots, a_s, p_s} \) on \( n \) vertices, let \( m, s \) and \( q \) be positive integers, and \( s \geq 3 \), then we have

(1) If \( p_i = m + 2(i - 1) \) for \( i = 1, 2, \ldots, s \), then there are no integers \( a_i \) \((i = 1, 2, \ldots, s)\) such that \( K_{a_1, p_1, a_2, p_2, \ldots, a_s, p_s} \) is an integral graph.

(2) If \( p_i = p_i q = [m+2(i-1)]q \) for \( i = 1, 2, \ldots, s \), let \( u_j = -(m+2j-3)q \) for \( j = 2, 3, \ldots, s \), then there are no integers \( a_i' \) \((i = 1, 2, \ldots, s)\) such that \( K_{a_1', p_1', a_2', p_2', \ldots, a_s', p_s} \) is an integral graph.

Proof. (1). Suppose that we can construct an integral graph \( K_{a_1, p_1, a_2, p_2, \ldots, a_s, p_s} \). From Theorem 2.6, we know that \( K_{a_1, p_1, a_2, p_2, \ldots, a_s, p_s} \) is integral if and only if there exist integers \( u_i \) and positive integers \( p_i \) \((i = 1, 2, \ldots, s)\) satisfying 

\[-p_s < u_s < -p_{s-1} < u_{s-1} < \cdots < u_2 < -p_1 < u_1 < +\infty \text{ such that all } a_k \text{ } (k = 1, 2, \ldots, s) \text{ are positive integers, where}

\[a_k = \frac{\prod_{i=1}^{s} (p_k + u_i)}{p_k \prod_{i=1, i \neq k} (p_k - p_i)} \text{ for } k = 1, 2, \ldots, s.\]

Hence, we can only choose

\[u_j = -(m+2j-3), \quad j = 2, 3, \ldots, s.\]

We obtain

\[
a_{s-1} = \frac{(m + 2s - 4 + u_1) \cdot (2s - 5)!!}{2(m + 2s - 4) \cdot (2s - 4)!!}, \quad (24)
\]

\[
a_s = \frac{(m + 2s - 2 + u_1) \cdot (2s - 3)!!}{(m + 2s - 2) \cdot (2s - 2)!!}, \quad (25)
\]
(m + 2s - 2) \cdot (2s - 2)!! \cdot a_s - 2(m + 2s - 4)(2s - 3) \cdot (2s - 4)!! \cdot a_{s-1} = 2 \cdot (2s - 3)!! \cdot a_s \cdot m^{s-1}. \quad (26)

Since s \geq 3, let d = ((m + 2s - 2 + u_1) \cdot (2s - 2)!! \cdot (m + 2s - 4)(2s - 3) \cdot (2s - 4)!!), then d = 2 \cdot (2s - 4)!! \cdot ((m + 2s - 2 + u_1)(s - 1), (m + 2s - 4)(2s - 3)). Thus, d \not| [2 \cdot (2s - 3)!!].

From Lemma 3.2, we know that there are no integral solutions (a_{s-1}, a_s) for Eqn.(26).

Hence, \( K_{a_1 p_1 a_2 p_2 \cdots a_s p_s} \) can't be an integral graph.

(2). Suppose that we can construct an integral graph \( K_{a'_1 p'_1 a'_2 p'_2 \cdots a'_s p'_s} \). From Theorem 2.6, we similarly obtain

\[
a'_{s-1} = \frac{[q(m + 2s - 4) + u'_1] \cdot (2s - 5)!!}{2q(m + 2s - 4) \cdot (2s - 4)!!}, \quad (27)
\]

\[
a'_s = \frac{[q(m + 2s - 2) + u'_1] \cdot (2s - 3)!!}{q(m + 2s - 2) \cdot (2s - 2)!!}, \quad (28)
\]

From (27) and (28), we have

\[
(m + 2s - 2) \cdot (2s - 2)!! \cdot a'_s - 2(m + 2s - 4)(2s - 3) \cdot (2s - 4)!! \cdot a'_{s-1} = 2 \cdot (2s - 3)!! \cdot a'_s. \quad (29)
\]

From Lemma 3.2, we know that there are no integral solutions (a'_{s-1}, a'_s) for Eqn.(29).

Hence, \( K_{a'_1 p'_1 a'_2 p'_2 \cdots a'_s p'_s} \) can't be an integral graph.

4 Further discussion

In this paper, we have mainly investigated integral complete r-partite graph \( K_{p_1 p_2 \cdots p_r} = K_{a_1 p_1, a_2 p_2, \ldots, a_s p_s} \) on n vertices. When \( s = 1, 2, 3 \), some results of such integral graphs can be found in [7, 8, 17] and the present paper. When \( s \geq 4 \), we have not found such integral graphs. We tried to get some general results. Thus, we raise the following questions for further study.

**Question 4.1.** Are there any integral complete r-partite graphs \( K_{p_1, \ldots, p_r} = K_{a_1 p_1, \ldots, a_s p_s} \) with arbitrarily large s?

For complete r-partite graphs \( K_{p_1 p_2, \ldots, p_r} = K_{a_1 p_1, a_2 p_2, \ldots, a_s p_s} \), when \( s = 1, 2, 3 \), let \( a_1 = a_2 = \cdots = a_s = 1 \), some results of such integral graphs can be found in [7, 8, 17] and the present paper. However, when \( s \geq 4 \), \( a_1 = a_2 = \cdots = a_s = 1 \), we have not found such integral graphs. Hence, we have

**Question 4.2.** Are there any integral complete r-partite graphs \( K_{p_1, \ldots, p_r} = K_{a_1 p_1, \ldots, a_s p_s} \) with \( a_1 = a_2 = \cdots = a_s = 1 \) when \( s \geq 4 \)?

For complete r-partite graphs \( K_{p_1 p_2, \ldots, p_r} = K_{a_1 p_1, a_2 p_2, \ldots, a_s p_s} \), we give a sufficient and necessary condition for \( K_{p_1 p_2, \ldots, p_r} = K_{a_1 p_1, a_2 p_2, \ldots, a_s p_s} \) to be integral. In particular, when \( s = 1, 2 \), we got all parameter solutions for \( K_{a_1 p_1, a_2 p_2} \) to be integral graphs in [7, 8] and the present paper. When \( s \geq 3 \), we haven't got such general good results. Hence, we have

**Question 4.3.** When \( s = 3, 4, 5, \ldots \), can we give a better sufficient and necessary condition for \( K_{a_1 p_1, a_2 p_2, \ldots, a_s p_s} \) to be integral?
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References


