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Removable Edges in Longest Cycles of 4-Connected Graphs*

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Abstract. Let G be a 4-connected graph. For an edge e of G, we do the following operations on G: first, delete the edge e from G, resulting the graph $G - e$; second, for all vertices x of degree 3 in $G - e$, delete x from $G - e$ and then completely connect the 3 neighbors of x by a triangle. If multiple edges occur, we use single edges to replace them. The final resultant graph is denoted by $G \ominus e$. If $G \ominus e$ is 4-connected, then e is called a removable edge of G. In this paper we obtain some results on removable edges in a longest cycle of a 4-connected graph G. We also show that for a 4-connected graph G of minimum degree at least 5 or girth at least 4, any edge of G is removable or contractible.

Key words. 4-Connected graph, Removable edge, Contractible edge, Edge-vertex-cut fragment

1. Introduction

All graphs considered here are finite and simple. For notations and terminology not given here, we refer to [3]. The concept of contractible edges and removable edges of graphs is a powerful tool to study the structures of graphs and to prove some properties of graphs by induction. In 1961, Tutte [12] gave the structural characterization for 3-connected graphs by using the existence of contractible edges and removable edges. He proved that every 3-connected graph with order at least 5 contains contractible edges. This is the earliest result concerning the concept of contractible edges and removable edges. A well-known application of the existence of contractible edges in 3-connected graphs was given by Thomssen [11]. By induction he gave a very simple unified proof for the three well-known theorems on planar graphs, i.e., the Kuratowski's Theorem: a graph is planar if and only if it does not contain any subgraph

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homeomorphic to K_5 or $K_{3,3}$; the Fary's Theorem: every planar graph has a plane linear representation; and the Tutte's Theorem: every 3-connected graph has a plane convex representation. The early proofs of the three theorems were very complicated and tedious. Another successful application of contractible edges is as follows. In 1974, Lovász posed the conjecture: let G be an n-connected graph and F be a set of independent edges of G. If n is even or $G - F$ is connected, then G has a cycle containing all the edges in F. Ando, Enomoto and Saito [2] showed that the conjecture is true for $n = 3$ by using contractible edges in 3-connected graphs. From the above examples we can see the importance of studying the existence and distribution of the contractible edges and removable edges of graphs.

Contractible edges and removable edges in 3-connected graphs have been studied extensively in literatures, see [1,2, 4–13], especially [6] for a survey on contractible edges. In this paper we shall focus on the study of removable edges in 4-connected graphs. First of all, we give the definition of a removable edge for a 4-connected graph. Let G be a 4-connected graph and e an edge of G. Consider the graph $G - e$ obtained by deleting the edge e from G. If $G - e$ has vertices of degree 3, we do the following operations on $G - e$. For all vertices x of degree 3 in $G - e$, delete x from $G - e$ and then completely connect the 3 neighbors of x by a triangle. If multiple edges occur, we use single edges to replace them. The final resultant graph is denoted by $G \ominus e$. Note that if there is no vertex of degree 3 in $G - e$, then $G \ominus e$ is simply the graph $G - e$.

Definition 1.1. For a 4-connected graph G and an edge e of G, if $G \ominus e$ is still 4-connected, then the edge e is called *removable*; otherwise, it is called *unremovable*. The set of all removable edges of G is denoted by $E_R(G)$; whereas the set of all unremovable edges of G is denoted by $E_N(G)$.

Definition 1.2. Let xy be an edge of a 4-connected graph G, and let G' be the simple graph obtained from G by first removing the edge xy, then identifying x and y by introducing a new vertex v_{xy} and finally making the new vertex v_{xy} be incident to all those edges (other than xy) that are originally incident to x or y. We call the edge xy *contractible* if G' is still 4-connected; otherwise, it is called *non-contractible*. The set of all contractible edges of G is denoted by $E_C(G)$. Let G be a 4-connected noncomplete graph, it is easy to see that an edge $e = xy$ is non-contractible if and only if there exists a vertex-cut of G containing x and y with 4 vertices.

The aim to introduce the concept of removable edges in 4-connected graphs is to find new method to construct 4-connected graphs and new method to prove some properties of 4-connected graphs. In [13], Yin proved that there always exist removable edges in 4-connected graphs G unless G is a 2-cyclic graph with order 5 or 6. There, he also showed that a 4-connected graph can be obtained from a 2-cyclic graph by the following four operations: (i) adding edges, (ii) splitting vertices, (iii) adding vertices and removing edges, and (iv) extending vertices.

Without specific statements, in the following G always denotes a 4-connected graph. The vertex set and edge set of G is denoted, respectively, by $V(G)$ and $E(G)$. The order and size of G is denoted, respectively, by $|G|$ and $|E(G)|$. For $x \in V(G)$, we simply write $x \in G$. The neighborhood of $x \in G$ is denoted by $\Gamma_G(x)$ and the degree of x is denoted by $d_G(x) = |\Gamma_G(x)|$. If no confusion, we simply write $d(x)$ for $d_G(x)$. The minimum degree of G is denoted by $\delta(G)$. If x and y are the two end-vertices of an edge e, we write $e = xy$. For a nonempty subset F of $E(G)$, or N of $V(G)$, the induced subgraph by F or N in G is denoted by $[F]$ or $[N]$. For $A, B \subset V(G)$ such that $A \neq \emptyset \neq B$ and $A \cap B = \emptyset$, define $[A, B] = \{xy \in E(G) \mid x \in A, y \in B\}$. If H is a subgraph of G, we say that G contains H. For a subset S of $V(G)$, $G - S$ denotes the graph obtained by deleting all the vertices in S from G together with all the incident edges. If $G - S$ is disconnected, we say that S is a vertex-cut of G. If $|S| = s$ for such an S, we say that S is an s-vertex-cut. The length of a shortest cycle of G is called the *girth* of G, which is denoted by $g(G)$. For $e \in E(G)$ and $S \subset V(G)$ such that $|S| = 3$, if $G - e - S$ has exactly two (connected) components, say A and B, such that $|A| \geq 2$ and $|B| \geq 2$, then we say that (e, S) is a *separating pair* and $(e, S; A, B)$ is a *separating group*, in which A and B are called the *edge-vertex-cut fragments*.

2. Background Knowledge

First of all, we list some known results on removable edges of 4-connected graphs, which can be found in [13].

Theorem 2.1. *Let G be a 4-connected graph with $|G| \geq 7$. An edge e of G is unremovable if and only if there is a separating pair (e, S), or a separating group $(e, S; A, B)$ in G.*

Theorem 2.2. *Let G be a 4-connected graph with $|G| \geq 8$ and let $(xy, S; A, B)$ be a separating group of G such that $x \in A$, $y \in B$ and $|A| \geq 3$. Then, every edge in $[\{x\}, S]$ is removable.*

Corollary 2.3. *Let G be a 4-connected graph with $|G| \geq 8$. Then, every triangle of G contains at least one removable edge.*

Theorem 2.4. *Let G be a 4-connected graph with $|G| \geq 7$. If for an unremovable edge xy, i.e., $xy \in E_N(G)$, there is a separating group $(xy, S; A, B)$, then all the edges in $E([S])$ are removable, i.e., $E([S]) \subset E_R(G)$.*

To end this section we introduce some more notations, which will be used later. Let $E_0 \subset E_N(G)$ such that $E_0 \neq \emptyset$ and let $(xy, S; A, B)$ be a separating group of G such that $x \in A$ and $y \in B$. If $xy \in E_0$, then A and B are called E_0-*edge-vertex-cut fragments*. An E_0-edge-vertex-cut fragment is called an E_0-*edge-vertex-cut end-fragment* of G if it does not contain any other E_0-edge-vertex-cut fragment of G as a proper subset. It is easy to see that any E_0-edge-vertex-cut fragment of G contains a such end-fragment.

3. Main Results

Before giving our main results, we first show the following results.

Theorem 3.1. *Let G be a 4-connected graph with $|G| \geq 8$ such that $\delta(G) \geq 5$ or $g(G) \geq 4$. Then, any edge of G is removable or contractible.*

Proof. By contradiction. Suppose there exists an edge e of G such that $e \in E_N(G)$ and $e \notin E_C(G)$, then we will deduce contradictions as follows.

Let $e = xy$. Since $e \notin E_C(G)$, there exists a 4-vertex-cut T of G such that $e \in [T]$. Let $G - T = C \cup D$ such that C is the union of at least one but not of all the component of $G - T$ and $D = G - T - C$. Since $e \in E_N(G)$, from Theorem 2.2 there exists a separating group $(e, S; A, B)$ of G such that $x \in A, y \in B$. It is easy to see that $x \in A \cap T, y \in B \cap T$. Let

$$X_1 = (C \cap S) \cup (T \cap S) \cup (A \cap T), \quad X_2 = (A \cap T) \cup (T \cap S) \cup (S \cap D),$$

$$X_3 = (D \cap S) \cup (T \cap S) \cup (B \cap T), \quad X_4 = (T \cap B) \cup (T \cap S) \cup (S \cap C).$$

From $|G| \geq 8$ we know that there exists a vertex $v \in V(G)$ satisfying $v \notin (S \cup T)$. Symmetrically, we may assume $v \in A \cap C$. Since $A \cap C \neq \emptyset$ and G is 4-connected, we have that X_1 is a vertex-cut of G, and so $|X_1| \geq 4$. Since $|X_1| + |X_3| = |S| + |T| = 7$, we have that $|X_3| \leq 3$. Since G is 4-connected, we have that $B \cap D = \emptyset$. We will discuss the following cases.

Case 1. If $B \cap C \neq \emptyset$, then X_4 is a vertex-cut of G, and so $|X_4| \geq 4$. From $|X_2| + |X_4| = |S| + |T| = 7$, we can get that $|X_2| \leq 3$. Since G is 4-connected, we have that $A \cap D = \emptyset$. Thus, $D = D \cap S$, and so $D \cap S \neq \emptyset$. Noticing that $|S| = 3$, we have that $|S \cap (C \cup T)| \leq 2$, and that $|X_1| \geq 4$ and $|X_4| \geq 4$. So, we have that $|A \cap T| \geq 2$ and $|B \cap T| \geq 2$. Noticing that $|T| = 4$, we have that $|A \cap T| = |B \cap T| = 2$ and $|S \cap T| = 0$. Thus, $|S \cap C| = 2$, and so $|S \cap D| = 1$, i.e., $|D| = 1$. Let $D = \{u\}$, then $xyux$ is a triangle of G, $d(u) = 4$ and $g(G) = 3$, a contradiction.

Case 2. If $B \cap C = \emptyset$, we have that $B = B \cap T$. From $|X_1| \geq 4$, we can get that $|S \cap C| \geq |B \cap T|$. From $|B| \geq 2$, we have that $|B \cap T| \geq 2$. If $|B \cap T| \geq 3$, noticing that $|T| = 4$, then $S \cap T = \emptyset$ and $A \cap T = \{x\}$. Based on the above argument, we have that $|S \cap C| \geq 3$. Noticing that $|S| = 3$, we have that $S \cap D = \emptyset$ and $|X_2| = 1$, and so $A \cap D = \emptyset$. Consequently, $D = \emptyset$, a contradiction. Therefore, $|B \cap T| = 2$. Noticing that $|S| = 3$, we have that $|S \cap (T \cup D)| \leq 1$. From $|B \cap T| = 2$ we can get that $|A \cap T| \leq 2$, and so $|X_2| \leq 3$. Therefore, $A \cap D = \emptyset$. Then $D = D \cap S$, and so

$D \cap S \neq \emptyset$. Noticing that $|S \cap C| \geq 2$ and $|S| = 3$, we have that $|D \cap S| = 1$. Let $D \cap S = \{u\}$, then $xyux$ is a triangle of G, $d(u) = 4$ and $g(G) = 3$, a contradiction. The proof is complete. □

From the proof of Theorem 3.1 we can get the following result.

Corollary 3.2. *Let G be a 4-connected graph with $|G| \geq 8$. If there exists an edge $e \in E(G)$ such that $e \in E_N(G)$ and $e \notin E_C(G)$, then $\delta(G) = 4$ and e is on a triangle of G.* □

Before showing our main result, the following lemma is needed.

Lemma 3.3. *Let G be a 4-connected graph with $|G| \geq 8$ and C be a longest cycle of G. Let $E(C) \subset E_N(G)$ and $E_0 = E(C)$. Suppose $x_1x_2 \in E(C)$ and $(x_1x_2, S; A, B)$ is a separating group such that $x_2 \in A, x_1 \in B$ and A is a E_0-edge-vertex-cut end-fragment. Then there are vertices $x, z, u, v \in V(C)$ such that $xz \in E(G)$(maybe $xz \notin E(C)$), $d(x) = d(z) = 4, \Gamma_G(x) \cap \Gamma_G(z) = \{u, v\}$ and $\{x, z\} \cap A \neq \emptyset$ and $\{x, z\} \cap B = \emptyset$.*

Proof. Since $E(C) = E_0$, the edge-vertex-cut fragment corresponding to any edge e on C is an E_0-edge-vertex-cut fragment. Take any edge x_1x_2 on C. Since $x_1x_2 \in E_N(G)$, from Theorem 2.1 there is a separating group $(x_1x_2, S; A, B)$ such that $x_2 \in A$ and $x_1 \in B$. It is easy to see that every E_0-edge-vertex-cut fragment contains a such end-fragment as a subset. Without loss of generality, from the arbitrariness of e on C we can assume that A is a such end-fragment. Since C is a cycle, we have that $(E(A) \cup [A, S]) \cap E(C) \neq \emptyset$. Take an edge x_2x_3 in the intersection. Since $x_2x_3 \in E(C) \subset E_N(G)$, take a separating group $(x_2x_3, S'; A', B')$ such that $x_2 \in A'$ and $x_3 \in B'$. Note that $x_2 \in A \cap A'$. Let

$$X_1 = (A' \cap S) \cup (S' \cap S) \cup (A \cap S'), \quad X_2 = (A \cap S') \cup (S' \cap S) \cup (S \cap B'),$$

$$X_3 = (B' \cap S) \cup (S' \cap S) \cup (B \cap S'), \quad X_4 = (S' \cap B) \cup (S' \cap S) \cup (S \cap A').$$

We distinguish the following cases to proceed the proof.

Case 1. $x_3 \in A \cap B'$ and $x_1 \in A' \cap B$.

Since $A' \cap B \neq \emptyset$, we know that X_4 is a vertex-cut of the graph $G - x_1x_2$. Since G is 4-connected and so $G - x_1x_2$ is 3-connected, we have that $|X_4| \geq 3$. Similarly, X_2 is a vertex-cut of $G - x_2x_3$ and so $|X_2| \geq 3$. Since $|X_2| + |X_4| = |S| + |S'| = 6$, we get that $|X_2| = |X_4| = 3$, and hence $|A' \cap S| = |A \cap S'|$ and $|B \cap S'| = |B' \cap S|$. Since $|S| = |S'| = 3$, we can distinguish the following four subcases for the value $|B \cap S'| = |B' \cap S|$.

Subcase 1.1. $|B \cap S'| = |B' \cap S| = 3$.

Note that $|S| = |S'| = 3$. This implies that $|X_1| = 0$. Hence, $\{x_1, x_3\}$ is a 2-vertex-cut of G, which contradicts to that G is 4-connected.

Subcase 1.2. $|B \cap S'| = |B' \cap S| = 2$.

We claim that $S \cap S' = \emptyset$. If not, since $|S| = |S'| = 3$, we get that $|S \cap S'| = 1$. So, $A' \cap S = A \cap S' = \emptyset$, and hence $|X_1| = 1$. Then, $X_1 \cup \{x_1, x_3\}$ would be a

3-vertex-cut of G, which contradicts to that G is 4-connected. Therefore, $S \cap S' = \emptyset$, and so $|A' \cap S| = |A \cap S'| = 1$. This implies that $|X_1| = 2$. We claim that $A \cap A' = \{x_2\}$. For otherwise, $|A \cap A'| \geq 2$. Since $|X_1| = 2$, it is easy to see that $\{x_2\} \cup X_1$ would be a 3-vertex-cut of G, a contradiction. Therefore, $A \cap A' = \{x_2\}$. Next, we claim that $A \cap B' = \{x_3\}$. Otherwise, $|A \cap B'| \geq 2$. Let $A_1 = A \cap B'$, $S_1 = X_2$ and $B_1 = G - x_2x_3 - S_1 - A_1$. Then, $(x_2x_3, S_1; A_1, B_1)$ is a separating group of G. Since $x_2x_3 \in E_0$, A_1 is an E_0-edge-vertex-cut fragment of G. Since $A_1 \subset A$, this contradicts to that A is an E_0-edge-vertex end-fragment. Therefore, $A \cap B' = \{x_3\}$. Let $A \cap S' = \{a\}$, $A' \cap S = \{b\}$ and $B' \cap S = \{u, v\}$. We claim $ab \in E(G)$. If not, then $\{u, v, x_2\}$ would be a 3-vertex-cut of G, which contradicts to that G is 4-connected. Therefore, $ab \in E(G)$. It is easy to see that $\Gamma_G(x_2) = \{x_1, x_3, a, b\}$ and $\Gamma_G(x_3) = \{a, u, v, x_2\}$. First, we let $e_1 = ab$, $S_1 = \{x_2\} \cup (B \cap S')$, $A_1 = A' \cap (B \cup S)$ and $B_1 = G - e_1 - S_1 - A_1$. Then, $(e_1, S_1; A_1, B_1)$ is a separating group of G, and so $ab \in E_N(G)$. Next, we claim $ax_3 \in E_R(G)$. If not, $ax_3 \in E_N(G)$, and hence there is a corresponding separating group $(ax_3, S_1; A_1, B_1)$ such that $a \in A_1$ and $x_3 \in B_1$. Since ax_2x_3a is a triangle of G, we have that $x_2 \in S_1$. Since $x_2x_3 \in E_N(G)$, from Theorem 2.2 we have that $|B_1| = 2$, say $B_1 = \{v_1, x_3\}$. Then, it is easy to see that $v_1x_2x_3v_1$ is a triangle of G and $v_1 \neq a$, which is impossible in G. Therefore, $ax_3 \in E_R(G)$. Since C is a cycle, $x_3 \in V(C)$ and $E(C) \subset E_N(G)$, we have that $\{x_3u, x_3v\} \cap E_N(G) \neq \emptyset$. Without loss of generality, we assume that $x_3u \in E_N(G)$.

Now we claim that $au \notin E(G)$.

By contradiction. Suppose $au \in E(G)$. Since $x_3u \in E_N(G)$, there is a corresponding separating group $(x_3u, T_1; C_1, D_1)$ such that $x_3 \in C_1$ and $u \in D_1$. So, $x_3 \in C_1 \cap B'$ and $u \in B' \cap D_1$. Since ax_3ua is a triangle of G, we have $a \in T_1$, and so $a \in S' \cap T_1$. Let

$$Y_1 = (A' \cap T_1) \cup (S' \cap T_1) \cup (C_1 \cap S'), \quad Y_2 = (C_1 \cap S') \cup (S' \cap T_1) \cup (B' \cap T_1),$$

$$Y_3 = (B' \cap T_1) \cup (S' \cap T_1) \cup (S' \cap D_1), \quad Y_4 = (D_1 \cap S') \cup (S' \cap T_1) \cup (A' \cap T_1).$$

(1) If $x_2 \in A' \cap C_1$, then Y_1 is a vertex-cut of $G - x_2x_3$. Since G is 4-connected, we have that $|Y_1| \geq 3$. Similarly, we have that $|Y_3| \geq 3$. Since $|Y_1| + |Y_3| = |S'| + |T_1| = 6$, we have that $|Y_1| = |Y_3| = 3$, and so $|A' \cap T_1| = |S' \cap D_1|$ and $|S' \cap C_1| = |B' \cap T_1|$. Since $a \in S' \cap T_1$ and $ab \in E_N(G)$, from Theorem 2.4 we have that $b \notin T_1 \cup S'$. Since $bx_2 \in E(G)$, we have $b \in A' \cap C_1$. From $x_3v \in E(G)$ we know that $v \notin D_1 \cap B'$, and so $v \in B' \cap (C_1 \cup T_1)$. Clearly, $|A' \cap T_1| = |S' \cap D_1| \leq 2$. We discuss the following cases.

(1.1) If $|A' \cap T_1| = |S' \cap D_1| = 2$, then noticing that $|T_1| = |S'| = 3$ and $a \in S' \cap T_1$ we have that $|S' \cap C_1| = |B' \cap T_1| = 0$. Since avx_3a is a triangle of G, we have $v \in B' \cap C_1$, and so $|B' \cap C_1| \geq 2$. Then, $\{a, x_3\}$ would be a 2-vertex-cut of G, a contradiction.

(1.2) If $|A' \cap T_1| = |S' \cap D_1| = 1$, then $|S' \cap T_1| \leq 2$. First, we claim that $B' \cap D_1 = \{u\}$. If not, then $|B' \cap D_1| \geq 2$. Since $\Gamma_G(a) = \{x_2, x_3, u, v, b\}$, by the foregoing arguments we have that $\Gamma_G(a) \cap (B' \cap D_1) = \{u\}$. So, $\{u\} \cup (Y_3 - \{a\})$ would be a 3-vertex-cut of G, a contradiction. Therefore, $B' \cap D_1 = \{u\}$. Let

$D_1 \cap S' = \{u_1\}$. If $|S' \cap T_1| = 1$, i.e., $S' \cap T_1 = \{a\}$, then $|Y_4| = 3$. Since G is 4-connected, we have that $D_1 \cap A' = \emptyset$. So, $u_1 \in \Gamma_G(a)$. However, it is easy to see that $u_1 \notin \{x_2, x_3, b, u, v\}$, a contradiction. Therefore, $|S' \cap T_1| = 2$ must hold. Then, $A' \cap D_1 = \emptyset$. If not, then $Y_4 - \{a\}$ would be a 3-vertex-cut of G, a contradiction. So, $A' \cap D_1 = \emptyset$ and it is easy to see that $au_1 \in E(G)$. However, this would imply that $u_1 \in \{b, u, v, x_2, x_3\}$, a contradiction.

(1.3) If $|A' \cap T_1| = |S' \cap D_1| = 0$, since D_1 is a connected subgraph of G, we have that $A' \cap D_1 = \emptyset$. From $|D_1| \geq 2$ we have that $|D_1 \cap B'| \geq 2$. Since $|Y_3| = |T_1| = 3$, by an analogous argument we have that $\Gamma_G(a) \cap (D_1 \cap B') = \{u\}$. So, $\{u\} \cup (Y_3 - \{a\})$ would be a 3-vertex-cut of G, a contradiction.

(2) If $x_2 \in A' \cap T_1$, then since $x_3x_2 \in E_N(G)$, from Theorem 2.2 we have that $|C_1| = 2$. Since C_1 is a connected subgraph of G, we have that $A' \cap C_1 = \emptyset$. If $S' \cap C_1 \neq \emptyset$, then due to $|C_1| = 2$, we have that $|S' \cap C_1| = 1$. From $a \in S' \cap T_1$ we have that $|D_1 \cap S'| \leq 1$. Since Y_3 is a vertex-cut of $G - x_3u$, we have that $|Y_3| \geq 3$, and so $|B' \cap T_1| \geq 1$. Noticing that $|T_1| = 3$, we get that $A' \cap T_1 = \{x_2\}$ and $|Y_4| = 3$. Since G is 4-connected, we have that $A' \cap D_1 = \emptyset$. Hence, $A' = \{x_2\}$, which contradicts to that $|A'| \geq 2$. If $S' \cap C_1 = \emptyset$, then $|B' \cap C_1| = 2$. Since $A' \cap T_1 \neq \emptyset$, we have that $|Y_2| = |T_1 \cap (B' \cup S')| \leq 2$, and so $\{x_3\} \cup Y_2$ would be a vertex-cut of G with cardinality less than 4, a contradiction.

By now we have proved that $au \notin E(G)$. Let $A_1 = \{a, x_2\}$, $S_1 = \{x_3\} \cup (S - \{u\})$ and $B_1 = G - x_1x_2 - S_1 - A_1$. Then, $(x_1x_2, S_1; A_1, B_1)$ is a separating group of G and $x_1x_2 \in E_0$. So, A_1 is an E_0-edge-vertex-cut fragment and $A_1 \subset A$, which contradicts to that A is an E_0-edge-vertex-cut end-fragment. Therefore, we conclude that Subcase 1.2 cannot occur.

Subcase 1.3. If $|B \cap S'| = |B' \cap S| = 1$, then we have that $|S \cap S'| \leq 2$. We discuss the following cases.

Subsubcase 1.3.1. If $|S \cap S'| = 2$, then $|A' \cap S| = |A \cap S'| = 0$ and so $|X_1| = 2$. We claim that $A \cap A' = \{x_2\}$. If not, then $|A \cap A'| \geq 2$, and so $X_1 \cup \{x_2\}$ would be a 3-vertex-cut of G, a contradiction. Hence, $A \cap A' = \{x_2\}$. Since $|X_2| = 3$, we claim that $A \cap B' = \{x_3\}$. Otherwise, $|A \cap B'| \geq 2$. Let $A_1 = A \cap B'$, $S_1 = X_2$ and $B_1 = G - x_2x_3 - S_1 - A_1$. Then, $(x_2x_3, S_1; A_1, B_1)$ is a separating group of G. Since $x_2x_3 \in E_0$, A_1 is an E_0-edge-vertex-cut fragment and $A_1 \subset A$, which contradicts to that A is an E_0-edge-vertex-cut end-fragment. Hence, $A \cap B' = \{x_3\}$. Then, we have that $d(x_2) = d(x_3) = 4$. Let $S \cap S' = \{a, b\}$. Obviously, we have that $ax_2, ax_3, bx_2, bx_3 \in E(G)$. Since C is a longest cycle of G, it is easy to see that $a, b \in V(C)$, and so Lemma 3.3 holds.

Subsubcase 1.3.2. If $|S \cap S'| = 1$, then $|A' \cap S| = |A \cap S'| = 1$. Since $|X_2| = 3$, by an argument analogous to that used in Subsubcase 1.3.1 we can show that $A \cap B' = \{x_3\}$. Let $A \cap S' = \{a\}$, $S \cap S' = \{b\}$, $B' \cap S = \{c\}$ and $S' \cap B = \{u\}$. Since $|X_3| = 3$ and G is 4-connected, we have that $B \cap B' = \emptyset$. Obviously, we have that $d(x_3) = d(c) = 4, x_3c \in E(G)$, $\Gamma_G(x_3) \cap \Gamma_G(c) = \{a, b\}$ and $\{x_3, c\} \cap A \neq \emptyset$. If $x_3c \in E(C)$, since ax_3ca and bx_3cb are triangles, we have $a, b \in V(C)$, and so

Lemma 3.3 holds. Hence, we may assume $x_3c \notin E(C)$. Then we have that $\{ax_3, bx_3\} \cap E(C) \neq \emptyset$, and so $c \in V(C)$. It suffices to prove $a, b \in V(C)$ for Lemma 3.3.

First we claim $a \in V(C)$. If not, then $ax_3, ac \notin E(C)$. Since $cx_3 \notin E(C)$ and $bx_3 \in E(C)$, we have $c \in V(C)$, and hence $bc, cu \in E(C)$. Noticing that $E(C) \subset E_N(G)$, we have $bc \in E_N(G)$. However, $bc \in E([S])$, from Theorem 2.4 we have $bc \in E_R(G)$, a contradiction. Therefore, $a \in V(C)$.

Next we claim $b \in V(C)$. If not, then $bx_3, bc \notin E(C)$. Since $c \in V(C)$, we have that $ac, cu, ax_3 \in E(C)$. Take a separating group $(ax_3, T'; C', D')$ such that $a \in C'$ and $x_3 \in D'$, then $c \in T'$. Since $\Gamma_G(c) = \{x_3, a, b, u\}$ and $au \notin E(G)$, we have that $b \in C'$ and $u \in D'$. Noticing $ac \in E_N(G)$, from Theorem 2.2 we have that $C' = \{a, b\}$, and so $ab \in E(G)$. Since C is a longest cycle of G, we have $b \in V(C)$, and so Lemma 3.3 holds.

Subsubcase 1.3.3. If $S \cap S' = \emptyset$, then we have that $|A \cap S'| = |A' \cap S| = 2$. Let $A \cap S' = \{a, b\}$, $B' \cap S = \{c\}$ and $B \cap S' = \{u\}$. Since $|X_3| = 2$, we have that $B \cap B' = \emptyset$. By an argument analogous to that used in Subsubcase 1.3.1 we can show that $A \cap B' = \{x_3\}$. Then, $\Gamma_G(x_3) = \{x_2, a, b, c\}$ and $\Gamma_G(c) = \{x_3, a, b, u\}$. If $cx_3 \in E(C)$, by an argument analogous to that used in Subsubcase 1.3.2 we can deduce Lemma 3.3. If $cx_3 \notin E(C)$, we may assume that $ax_3 \in E(C)$, then $c \in V(C)$. We claim $b \in V(C)$. If not, then $ac, uc \in E(C)$. Take a separating group $(ax_3, T_1; C_1, D_1)$ such that $x_3 \in C_1$ and $a \in D_1$, then $c \in T_1$. Since $ac \in E_N(G)$, from Theorem 2.2 we have that $|D_1| = 2$. Since $\Gamma_G(c) = \{x_3, a, b, u\}$, we have that $b \in D_1$ or $u \in D_1$. If $u \in D_1$, then $au \in E(G)$, contradicting to that $a \in A$ and $u \in B$; if $b \in D_1$, then $ab \in E(G)$, and so $b \in V(C)$, a contradiction. Hence, $b \in V(C)$. Since $d(x_3) = d(c) = 4$ and $\Gamma_G(x_3) \cap \Gamma_G(c) = \{a, b\}$, our Lemma 3.3 holds.

Subcase 1.4. If $B \cap S' = \emptyset = B' \cap S$, then $B \cap B' = \emptyset$, B' is an E_0-edge-vertex-cut fragment and $B' \subset A$, which contradicts to that A is an E_0-edge-vertex-cut end-fragment.

Case 2. $x_3 \in B' \cap S$ and $x_1 \in A' \cap B$.

By an argument analogous to that used in Subsubcase 1.3.2 we can deduce that $A \cap A' = \{x_2\}, A \cap S' = \{a\}, S \cap A' = \{b\}, S \cap B' = \{x_3, u\}$ and $S \cap S' = \emptyset$. Hence, $d(x_2) = d(a) = 4, ax_2 \in E(G)$ and $\Gamma_G(x_2) \cap \Gamma_G(a) = \{b, x_3\}$. Since $x_2x_3 \in E(C)$ and ax_2x_3a is a triangle, we have $a \in V(C)$. We claim $b \in V(C)$. If not, then $bx_2, ba \notin E(C)$, and so $ax_3, au \in E(C)$. Take a separating group $(ax_3, T'; C', D')$ such that $a \in C'$ and $x_3 \in D'$, then $x_2 \in T'$. Since $\Gamma_G(x_2) = \{x_3, a, b, x_1\}$ and $ab \in E(G)$, we have that $b \in C'$ and $x_1 \in D'$. Noticing $x_2x_3 \in E_N(G)$, from Theorem 2.2 we have that $C' = \{x_3, x_1\}$, and so $x_1x_3 \in E(G)$, contradicting to that $x_1 \in A'$ and $x_3 \in B'$. Therefore, $b \in V(C)$, and so Lemma 3.3 holds.

Case 3. $x_3 \in A \cap B'$ and $x_1 \in B \cap S'$.

By an argument analogous to that used in Case 2 we can deduce that $A \cap A' = \{x_2\}, S \cap A' = \{b\}, A \cap S' = \{a\}$ and $S \cap S' = \emptyset$. Since A is an E_0-edge-vertex-cut end-fragment, by an argument analogous to that used in Subsubcase 1.3.1 we can deduce that $A \cap B' = \{x_3\}$. Hence, $d(x_2) = d(b) = 4$ and $\Gamma_G(x_2) \cap \Gamma_G(b) = \{a, x_1\}$. Since $x_1x_2, x_2x_3 \in E(C)$ and bx_1x_2b and ax_2x_3a are triangles, we have $a, b \in V(C)$, and so Lemma 3.3 holds.

Case 4. $x_3 \in B' \cap S$ and $x_1 \in B \cap S'$.

By an argument analogous to that used in Case 2 we can deduce that $A \cap A' = \{x_2\}, S \cap A' = \{b\}, A \cap S' = \{a\}$ and $S \cap S' = \emptyset$. Hence, $d(x_2) = d(b) = 4$ and $\Gamma_G(x_2) \cap \Gamma_G(b) = \{a, x_1\}$. Since $x_1x_2, x_2x_3 \in E(C)$, and bx_1x_2b and ax_2x_3a are triangles, we have $a, b \in V(C)$, and so Lemma 3.3 holds. By now, our proof is complete. □

Before proceeding, we introduce the following notations.

Definition 3.4. *Let G be a 4-connected graph and H be a subgraph of G. If $V(H) = \{u, v, x, z\}$, $E(H) = \{xz, ux, vx, uz, vz\}$ and $d(x) = d(z) = 4$, then H is called a bi-triangle, and x, z are called its inner vertices. If a cycle C of G contains the vertices u, v, x and z, we say that C passes through the bi-triangle H.*

From Lemma 3.3 we can directly deduce the following result.

Corollary 3.5. *Let G be a 4-connected graph with $|G| \geq 8$. If a longest cycle C of G does not pass through any bi-triangle, then C contains at least one removable edge.*

In fact, we can obtain the following two results.

Theorem 3.6. *Let G be a 4-connected graph with $|G| \geq 8$. If a longest cycle C of G does not pass through any bi-triangle, then C contains at least two removable edges.*

Proof. By contradiction. Suppose C contains at most one removable edge of G. Let $F = E(C) \cap E_R(G)$. Then, $|F| \leq 1$. Let $E_0 = E(C) - F$, and so $E_0 \neq \emptyset$. Then, for an edge uw in E_0, there is a separating group $(uw, S'; A', B')$ of G such that $u \in A'$ and $w \in B'$. Since $|F| \leq 1$, we have that $(E(A') \cup [A', S']) \cap F = \emptyset$, or $(E(B') \cup [B', S']) \cap F = \emptyset$. Without loss of generality, we assume that $(E(A') \cup [A', S']) \cap F = \emptyset$. Since A' is an E_0-edge-vertex-cut fragment, there must exist an E_0-edge-vertex-cut end-fragment contained in A', say A. Then, corresponding to A there is a separating group $(xy, S; A, B)$ of G such that $x \in A$, $y \in B$, $|S| = 3$ and $xy \in E_0$. Obviously, $(E(A) \cup [A, S]) \cap F = \emptyset$. Since C is a cycle of G, there exits an edge $xz \in E(C) \cap (E(A) \cup [A, S]) \neq \emptyset$. Analogously, we take the separating group $(xz, S'; A', B')$ of G such that $x \in A', z \in B'$. By analogous arguments used in the proof of Lemma 3.3 we can show that C passes through at least one bi-triangle, which contradicts to the assumption of the theorem. The proof is complete. □

Theorem 3.7. *Let G be a 4-connected graph with $|G| \geq 8$. If a longest cycle C of G passes through at most one bi-triangle, then C contains at least one removable edge.*

Proof. By contradiction. Suppose C does not contain any removable edge. Let $E_0 = E(C)$. If C does not pass through any bi-triangle, then from Theorem 3.6 the theorem holds. So, next we assume that C passes through only one bi-triangle H as defined in Definition 3.4. We take an E_0-edge-vertex-cut end-fragment A and its corresponding separating group $(ww', S; A, B)$ of G such that $w \in A$ and $w' \in B$. If $xz = ww'$, we may assume that $x = w, z = w'$, then from Lemma 3.3 there exists an inner vertex $x' \in A$ and $x' \in V(C)$. Let H' be the bi-triangle containing x' as its an inner vertex. From Lemma 3.3 we know that C passes through H'. It is easy to see that $V(H') \subset V(A) \cup V(S)$, and so $z \notin V(H')$ and $H \neq H'$, which contradicts to that C passes through only one bi-triangle. If $xz \neq ww'$, from Lemma 3.3 we have that $V(H) \subset V(A) \cup V(S)$, and so $V(H) \cap V(B) = \emptyset$, then B must contain an E_0-edge-vertex-cut end-fragment B' satisfying that $V(B') \cap V(H) = \emptyset$. Since B' is an E_0-edge-vertex-cut end-fragment, from Lemma 3.3 there are vertices $x', z', u', v' \in V(C)$ such that $x'z' \in E(G)$, $d(x') = d(z') = 4, \Gamma_G(x') \cap \Gamma_G(z') = \{u', v'\}$ and $\{x', z'\} \cap B' \neq \emptyset$, i.e., C passes through the bi-triangle H' and $H' \neq H$, which contradicts to that C passes through only one bi-triangle. The proof is complete. □

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References

1. Andersen, L.D., Fleischner, H., Jackson, B.: Removable edges in cyclically 4-edge-connected cubic graphs. Graphs Comb. **4**, 1–21 (1988)
2. Ando, K., Enomoto, H., Saito, A.: Contractible edges in 3-connected graphs, J. Comb. Theory, Ser.B **42**, 87–93 (1987)
3. Bondy, J.A., Murty, U.S.R.: Graph Theory with Applications, North-Holland, Amsterdam 1976
4. Holton, D.A., Jackson, B., Saito, A., Wormald, N.C.: Removable edges in 3-connected graphs. J. Graph Theory **14**, 465–473 (1990)
5. Huck, A.: About the number of removable edges in n-regular n-edge-connected graphs. Contemporary Methods in Graph Theory, pp. 405–408, Bibliographisches Institute Mannheim 1990
6. Kriesell, M.: A survey on contractible edges in graphs of a prescribed vertex connectivity. Graphs Comb. **18**, 1–30 (2002)
7. McCuaig, W.D., Haglin, D.J., Venkatesan, S.M.: Contractible edges in 4-connected maximal planar graphs. Ars Comb. **31**, 199–203 (1991)
8. Ou, J.P., Su, J.J.: The number of removable edges in 3-connected graphs (in Chinese), Math. Appl. **14**, 80–84 (2001)
9. Su, J.J.: The removable edges in 3-connected graphs. J. Comb. Theory, Ser. B **75**, 74–87 (1999)
10. Su, J.J.: Removable edges of cycles in 3-connected graphs (in Chinese). Chinese Science Bulletin **44**, 921–926 (1999)
11. Thomassen, C.: Planarity and duality of finite graphs. J. Comb. Theory, Ser. B **29**, 224–271 (1980)
12. Tutte, W.T.: A theorey of 3-connected graphs, Indag. Math. **23**, 441–455 (1961)
13. Yin, J.H.: Removable edges in 4-connected graphs and the structures of 4-connected graphs. J. Syst. Sci. Math. Sci. **19**, 434–438 (1999)

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