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NOTE ON REGULAR D-OPTIMAL MATRICES**

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Abstract

Let A be a $j \times d$ $(0,1)$ matrix. It is known that if $j = 2k-1$ is odd, then $\det(AA^T) \le (j+1)((j+1)d/4j)^j$; if j is even, then $\det(AA^T) \le (j+1)((j+2)d/4(j+1))^j$. A is called a regular D-optimal matrix if it satisfies the equality of the above bounds. In this note, it is proved that if $j = 2k-1$ is odd, then A is a regular D-optimal matrix if and only if A is the adjacent matrix of a $(2k-1, k, (j+1)d/4j)$-BIBD; if $j = 2k$ is even, then A is a regular D-optimal matrix if and only if A can be obtained from the adjacent matrix B of a $(2k+1, k+1, (j+2)d/4(j+1))$-BIBD by deleting any one row from B. Three 21×42 regular D-optimal matrices, which were unknown in [11], are also provided.

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§1. Introduction

Let $M_{j,d}(0,1)$ be the set of all $j \times d$ $(0,1)$ matrices. The problem of finding the maximum value of $\det(AA^T)$ for all $A \in M_{j,d}(0,1)$ has received considerable attention over the past decade primarily for its significance in finding a j-simplex of the maximum volume in the d-dimensional unit cube and in statistical design theory[6,7].

The matrices $A \in M_{j,d}(0,1)$ such that $\det(AA^T)$ is maximum are called D-optimal matrices. Few results are known for D-optimal matrices. In [10, 11], Neubauer, Watkins and Zeitlin proved that for $A \in M_{j,d}(0,1)$, if $j = 2k-1$ is odd, then $\det(AA^T) \le (j+1)((j+1)d/4j)^j$; if $j = 2k$ is even, then $\det(AA^T) \le (j+1)((j+2)d/4(j+1))^j$. They defined that a D-optimal matrix A is regular if it satisfies the equality of the above bounds. Some new infinitely families of regular D-optimal matrices are constructed by Hadamard matrices and supplementary difference sets[11].

The purpose of this note is to show that if $j = 2k-1$, then a matrix $A \in M_{j,d}(0,1)$ is a regular D-optimal matrix if and only if A is the adjacent matrix of a $(2k-1, k, (j+1)d/4j)$-BIBD (the definition of BIBD can be seen in Definition 2.2 below); if $j = 2k$, then a $(0,1)$ matrix $A \in M_{j,d}(0,1)$ is a regular D-optimal matrix if and only if A can be obtained from the adjacent matrix B of a $(2k+1, k+1, (j+2)d/4(j+1))$-BIBD by deleting any one row

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from B. We also provide three 21×42 regular D-optimal matrices, which were unknown in [11].

§2. Preliminaries

We begin this section with some definitions on design theory and some relative results.

Definition 2.1. *Let v, λ be positive integers and $K \subseteq N$ be a set of positive integers. A (v, K, λ) pairwise balanced design, denoted by (v, K, λ)-PBD, is a finite collection $\mathbf{B} = \{B_1, B_2, \cdots, B_b\}$ of subsets of $X = \{1, 2, \cdots, v\}$ such that*

(1) *each pair of elements $i, j \in X$ occurs in exactly λ blocks in $\mathbf{B}$, and*

(2) *for each $B_i \in \mathbf{B}$, $|B_i| \in K$.*

Definition 2.2. *Let v, k, λ be positive integers with $k < v$. A balanced incomplete block design, denoted by (v, k, λ)-BIBD, is a finite collection $\mathbf{B} = \{B_1, B_2, \cdots, B_b\}$ of subsets of $\{1, 2, \cdots, v\}$ such that*

(1) *each B_j has cardinality k, and*

(2) *each pair $i, j \in \{1, 2, \cdots, v\}$ occurs in exactly λ subsets in $\mathbf{B}$.*

Obviously, a BIBD is a special PBD.

It is an elementary result in block design theory that if $\mathcal{B} = (X, \mathbf{B})$ is a (v, k, λ)-BIBD, then each element $i \in \{1, 2, \cdots, v\}$ occurs in the same number r of subsets in $\mathbf{B}$ and that $b = |\mathbf{B}|$ and r satisfies the conditions

$$\lambda(v-1) = r(k-1), \quad bk = vr. \tag{2.1}$$

Thus b and r are determined by the other three parameters v, k, λ of the design. We sometimes refer to a (v, k, λ)-BIBD as a (v, b, r, k, λ)-BIBD.

The incidence matrix $A = (a_{ij})$ of a (v, b, r, k, λ)-BIBD $\mathcal{B} = (X, \mathbf{B})$ is defined by

$$a_{ij} = \begin{cases} 1, & \text{if } \ i \in B_j, \\ 0, & \text{otherwise.} \end{cases}$$

It is well known that a $(0, 1)$ matrix A is the incidence matrix of a (v, b, r, k, λ)-BIBD if and only if the following hold

$$AA^T = (r - \lambda)I_v + \lambda J_v, \tag{2.2}$$

$$J_v A = kJ_{v,b}, \tag{2.3}$$

where I_v denotes the identity matrix of order v and $J_{v,b}$ denotes $v \times b$ matrix with all its entries 1.

Now we mention the known upper bounds for $\det(AA^T)$ separated into the cases $-$ j odd and j even.

Lemma 2.1.[11] *If $j = 2k - 1$ is odd and $A \in M_{j,d}(0, 1)$, then*

$$\det(AA^T) \le (j+1)\left(\frac{(j+1)d}{4j}\right)^j.$$

Equality holds if and only if

(1) *$AA^T = t(I + J)$ for some integer t, and either of the following conditions are met:*

(2a) *each column of A contains exactly k ones*

or

(2b) $t = (j+1)d/4j$.

Lemma 2.2.[11] *If $j = 2k$ is even and $A \in M_{j,d}(0,1)$, then*

$$\det(AA^T) \le (j+1)\Big(\frac{(j+2)d}{4(j+1)}\Big)^j.$$

Equality holds if and only if

(1) $AA^T = t(I+J)$ *for some integer t, and either of the following conditions are met:*

(2a) *each column of A contains either k or $k+1$ ones*

or

(2b) $t = (j+2)d/4(j+1)$.

Lemma 2.3.[11] (1) *Assume that $j = 2k-1$ is odd and $A \in M_{j,d}(0,1)$ contains a column with fewer than k or more than k ones. Then*

$$\det(AA^T) \le \Big(\frac{1}{j+1}\Big)^{j-1}\Big(\frac{k^2d-1}{j}\Big)^j. \tag{2.4}$$

(2) *Assume that $j = 2k$ and $A \in M_{j,d}(0,1)$ contains a column with fewer than k or more than $k+1$ ones. Then*

$$\det(AA^T) \le \Big(\frac{1}{j+1}\Big)^{j-1}\Big(\frac{k(k+1)d-2}{j}\Big)^j. \tag{2.5}$$

Lemma 2.4. *Suppose $\mathcal{B} = (X, \mathbf{B})$ is a $(v, \{k, k+1\}, \lambda)$-PBD satisfying*

(1) *each point of X occurs in exactly r blocks;*

(2) *there are r blocks of size k.*

Then, by adding a new point y to X and to all size k blocks, we obtain a $(v+1, k+1, \lambda)$-BIBD.

Proof. Let $x \in X$ and suppose that there are a_x and b_x blocks of size $k+1$ and k respectively, containing x. It is obvious that

$$a_x + b_x = r. \tag{2.6}$$

Considering all the pairs containing x, we have

$$ka_x + (k-1)b_x = \lambda(v-1). \tag{2.7}$$

From (2.6) and (2.7), we have

$$b_x = rk - \lambda(v-1). \tag{2.8}$$

Therefore, b_x is a constant number independent of x, and so is a_x.

Let $|\mathbf{B}| = b$. There are $b-r$ blocks of size $k+1$. From size $k+1$ blocks, we get $(b-r)\binom{k+1}{2}$ pairs. From size k blocks, we get $r\binom{k}{2}$ pairs. Hence we have

$$(b-r)\binom{k+1}{2} + r\binom{k}{2} = \lambda\binom{v}{2}. \tag{2.9}$$

Since each point of X appears in exactly r blocks, we get

$$rk + (b-r)(k+1) = vr. \tag{2.10}$$

From (2.9) and (2.10), we get

$$rk = \lambda v. \tag{2.11}$$

Further from (2.8), we get

$$b_x = \lambda.$$

This means that each point $x \in X$ appears in λ blocks of size k. Thus for each $x \in X$, the pair {x, y} occurs in exactly λ blocks. The conclusion then follows.

§3. Main Results

Theorem 3.1. *Let $j = 2k-1$ be an odd integer and $A \in M_{j,d}$. Then the following statements are equivalent:*

(1) *A is a regular D-optimal matrix, i.e.,* $\det(AA^T) = (j+1)((j+1)d/4j)^j$.

(2) *A is the adjacent matrix of a $(2k-1, k, (j+1)d/4j)$-BIBD.*

Proof. (1) $\Rightarrow$ (2). A is the adjacent matrix of a $(2k-1,k,(j+1)d/4j)$-BIBD. By (2.2), $AA^T = (j+1)d/4j(I_j+J_j)$, it follows that $\det(AA^T) = (j+1)((j+1)d/4j)^j$, i.e., A is a regular D-optimal matrix.

(2) $\Rightarrow$ (1). Now we assume that A is a regular D-optimal matrix, i.e., $\det(AA^T) = (j+1)((j+1)d/4j)^j$. By Lemma 2.1, we have

$$AA^T = t(I+J) \text{ for some positive integer } t.$$

We prove that each column of A contains exactly k ones and that $t = (j+1)d/4j$.

First we prove that each column of A contains exactly k ones.

Suppose, on the contrary, A contains a column with fewer than k ones or more than k ones. Then by Lemma 2.3,

$$\det(AA^T) \le \Big(\frac{1}{j+1}\Big)^{j-1}\Big(\frac{k^2d-1}{j}\Big)^j.$$

Since

$$\begin{aligned}
&(j+1)\Big(\frac{(j+1)d}{4j}\Big)^j - \Big(\frac{1}{j+1}\Big)^{j-1}\Big(\frac{k^2d-1}{j}\Big)^j \\
&= \frac{(j+1)^{(j+1)}d^j(j+1)^{(j-1)} - (k^2d-1)^j4^j}{(4j)^j(j+1)^{j-1}} \\
&= \frac{(j+1)^{2j}d^j - ((\frac{j+1}{2})^2d-1)^j4^j}{4^jj^j(j+1)^{j-1}} \\
&> \frac{(j+1)^{2j}d^j - ((\frac{j+1}{2})^2d)^j4^j}{4^jj^j(j+1)^{j-1}} \\
&= 0,
\end{aligned}$$

$\det(AA^T) < (j+1)((j+1)d/4j)^j$. This contradicts the assumption that A is a regular D-optimal matrix. Thus each column of A contains exactly k ones. By (2.2) and (2.3), A is an adjacent matrix of some $(2k-1,k,t)$-BIBD with $r = 2t$.

Now we prove that $t = (j+1)d/4j$. By (2.1), we have $2t(2k-1) = dk$. It follows that

$$2t = \frac{dk}{2k-1} = \frac{d(j+1)}{2j}.$$

Thus $t = d(j+1)/4j$. So A is the adjacent matrix of a $(2k-1,k,(j+1)d/4j)$-BIBD and Theorem 3.1 is now proved.

Theorem 3.2. *Let $j = 2k$ be even and $A \in M_{j,d}(0,1)$. Then the following statements are equivalent:*

(1) *A is a regular D-optimal matrix, i.e.,* $\det(AA^T) = (j+1)((j+2)d/4(j+1))^j$.

(2) *A is the adjacent matrix of a $(j,\{k,k+1\},(j+2)d/4(j+1))$- PBD, say $\mathcal{B}=(X,\mathbf{B})$, which satisfies the following conditions:*

(2a) *each point of X appears in exactly $(j+2)d/2(j+1)$ blocks; and*

(2b) *there are $(j+2)d/2(j+1)$ blocks of size k.*

(3) *A is a matrix obtained from the adjacent matrix B of a $(2k+1,k+1,(j+2)d/4(j+1))$-BIBD by deleting any one row from B.*

Proof. (1) $\Rightarrow$ (2). Assume that A is a regular D-optimal matrix, i.e., $\det(AA^T)=(j+1)((j+2)d/4(j+1))^j$. By Lemma 2.2,

$$AA^T=t(I_j+J_j)\quad\text{for some positive integer } t.$$

In order to prove that A is the adjacent matrix of a $(j,\{k,k+1\},(j+2)d/4(j+1))$-PBD which satisfies the conditions (2a) and (2b), we first prove that each column of A contains exactly k or $k+1$ ones.

Suppose, on the contrary, that A contains a column with fewer than k or more than $k+1$ ones. By Lemma 2.3, we have

$$\det(AA^T)\le\Big(\frac{1}{j+1}\Big)^{j-1}\Big(\frac{k(k+1)d-2}{j}\Big)^j.$$

Since

$$\begin{aligned}&(j+1)\Big(\frac{(j+2)d}{4(j+1)}\Big)^j-\Big(\frac{1}{j+1}\Big)^{j-1}\Big(\frac{k(k+1)d-2}{j}\Big)^j\\ &=\frac{j^j(j+2)^jd^j-(\frac{j(j+2)}{4}d-2)^j4^j}{4^j(j+1)^{j-1}j^j}\\ &>\frac{j^j(j+2)^jd^j-(j(j+2))^jd^j}{4^j(j+1)^{j-1}j^j}\\ &=0,\end{aligned}$$

$\det(AA^T)<(j+1)((j+2)d/4(j+1))^j$. This contradicts the assumption that A is a regular D-optimal matrix. Thus each column of A contains either k or $k+1$ ones.

Thus A is the adjacent matrix of a $(j,\{k,k+1\},t)$-PBD. Let the PBD be $\mathcal{B}=(X,\mathbf{B})$. Since $AA^T=t(I_j+J_j)$, each point in X appears in exactly $2t$ blocks.

Now we determine the parameter t and show that there are $(j+2)d/2(j+1)$ blocks of size k.

Assume that there are m blocks in $\mathbf{B}$ of size k. From the blocks of size k, we get $m\binom{k}{2}$ pairs, and from the blocks of size $k+1$, we get $(d-m)\binom{k+1}{2}$ pairs. So

$$mk+(d-m)(k+1)=2tj,$$
$$m\binom{k}{2}+(d-m)\binom{k+1}{2}=t\binom{j}{2}.$$

It follows that $t=(j+2)d/4(j+1)$ and $m=(j+2)d/2(j+1)$. Thus A is the adjacent matrix of a $(j,\{k,k+1\},(j+2)d/4(j+1))$-PBD which satisfies the conditions (2a) and (2b).

(2)$\Rightarrow$ (3). By Lemma 2.4.

(3)$\Rightarrow$ (1). A is a matrix obtained from the adjacent matrix, say $\mathcal{B}$, of a $(j+1,k+1,(j+2)d/4(j+1))$-BIBD, then $BB^T=(j+2)d/4(j+1)(I_{j+1}+J_{j+1})$. It follows that $AA^T=(j+2)d/4(j+1)(I_j+J_j)$. Thus $\det(AA^T)=(j+1)((j+2)d/4(j+1))^j$, i.e., A is a regular D-optimal matrix. Theorem 3.2 is now proved.

Now we give three D-optimal 21×42 optimal matrices, which were unknown for the authors in [11] (see [11, p.115]). In [3], the authors gave three $(21,42,20,10,9)$-BIBD. The initial blocks of three solutions are given by

$$\begin{aligned} D_1: \quad & A_1 = (0,1,2,4,5,8,9,12,14,20), \\ & B_1 = (0,5,8,10,12,14,15,18,19,20); \\ D_2: \quad & A_2 = (0,1,4,5,7,8,9,10,17,18), \\ & B_2 = B_1; \\ D_3: \quad & A_3 = (0,1,2,4,6,7,9,10,17,18), \\ & B_3 = (0,4,5,7,11,13,14,15,16,19). \end{aligned}$$

The supplementary design of the three BIBDs are $(21,42,22,11,11)$-BIBDs. By Theorem 3.1, we get three 21×42 regular D-optimal matrices.

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