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w-DENSITY AND w-BALANCED PROPERTY OF WEIGHTED GRAPHS

Zhang Shenggui[1] Sun Hao[1] Li Xueliang[2]

Abstract. The notion of w-density for the graphs with positive weights on vertices and nonnegative weights on edges is introduced. A weighted graph is called w-balanced if its w-density is no less than the w-density of any subgraph of it. In this paper, a good characterization of w-balanced weighted graphs is given. Applying this characterization, many large w-balanced weighted graphs are formed by combining smaller ones. In the case where a graph is not w-balanced, a polynomial-time algorithm to find a subgraph of maximum w-density is proposed. It is shown that the w-density theory is closely related to the study of SEW(G,w) games.

§ 1 Introduction

All graphs considered in this paper are finite undirected ones without loops or multiple edges. Our terminology and notation are standard except as indicated. A good reference for any undefined terms is [1].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *density* of G is defined by

$$d(G) = \frac{\varepsilon(G)}{\nu(G)},$$

where $\nu(G)$ and $\varepsilon(G)$ denote $|V(G)|$ and $|E(G)|$, respectively. G is said to be *balanced* if for each subgraph H of G we have $d(H) \leqslant d(G)$, where $V(H)$ is assumed to be nonempty. If G is not balanced, then it contains a subgraph with greater density than that of G. In particular, we use $m(G)$ to denote the maximum density of a subgraph of G, i.e.

$$m(G) = \max_{H\subseteq G} d(H).$$

$m(G)$ will be called the *global density* of G.

The notions of density and balanced graphs originated in the work of Erdős and Rényi on random graphs[5], in a result giving the probability that a random graph contains a giv-

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en graph. Subsequently, many other applications of density and balanced graphs to random graphs and other areas appeared. For more details, see[6].

The graph G is called a *weighted graph* if there exist a vertex-weight function $w^V: V(G)\to R^+$ and an edge-weight function $w^E: E(G)\to R_0^+$, where R^+ refers to the set of all positive real numbers and R_0^+ the set of all nonnegative real numbers. For a subgraph H of G, the *vertex-weight* and the *edge-weight* of H are defined by

$$w^V(H) = \sum_{v\in V(H)} w^V(v) \text{ and } w^E(H) = \sum_{e\in E(H)} w^E(e),$$

respectively. The *w-density* of G is defined by

$$wd(G) = \frac{w^E(G)}{w^V(G)}.$$

A weighted graph G is called *w-balanced* if for each subgraph H of G, we have $wd(H)\leqslant wd(G)$, where $V(H)$ is assumed to be nonempty. So, when to check whether a given graph G is w-balanced, it suffices to verify that $wd(H)\leqslant wd(G)$ for any induced subgraph H of G. If G is not w-balanced, then there exists a subgraph with greater w-density than that of G. Let $wm(G)$ denote the maximum w-density of a subgraph of G, i. e.,

$$wm(G) = \max_{H\subseteq G} wd(H).$$

$wm(G)$ will be called the *global w-density* of G.

An unweighted graph can be regarded as a weighted graph in which for each vertex v and each edge e we assign weight $w^V(v)=w^E(e)=1$. Thus in an unweighted graph, $wd(G)=d(G)$ and $wm(G)=m(G)$.

The purpose of this paper is to generalize some results on density and balanced property of unweighted graphs to weighted graphs. In § 2 we prove a good characterization of w-balanced weighted graphs and use this characterization to form large w-balanced weighted graphs from smaller ones. In § 3 we propose an efficient algorithm to find a subgraph with maximum w-density in the case where G is not w-balanced. In § 4 we show that our results lead to an efficient polynomial algorithm to determine whether an imputation is a member of the core of an SEW(G, w) game, when all the weights of the edges in the corresponding graph G are nonnegative.

§ 2 w-balanced weighted graphs

This section is devoted to giving a good characterization to w-balanced weighted graphs and to combining w-balanced weighted graphs to form larger ones. Similar results on unweighted graphs can be found in a paper of Penrice[7]. Although the proofs of the results in this section are similar to those of unweighted graphs, we include most of the proofs here for the convenience of readers.

2.1 A characterization of w-balanced weighted graphs

Suppose that G is a weighted graph with vertex-weight function w^V and edge-weight funcion w^E. We define the *incidence graph* $I(G)$ to be the bipartite graph with independent sets $X=E(G)$ and $Y=V(G)$ and with $e\in X$ adjacent to $v\in Y$ if and only if v is an end-vertex of e in G. A *normalized edge weight assignment* is an assignment of nonnegative real weights to the edges of $I(G)$ such that

1. For every $e\in X$, the sum of the weights of the edges of $I(G)$ incident to e is $w^E(e)w^V(G)$;

2. For every $v\in Y$, the sum of the weights of the edges of $I(G)$ incident to v is $w^V(v)w^E(G)$.

Theorem 1. Let G be a weighted graph, then G is w-balanced if and only if $I(G)$ has a normalized edge weight assignment.

Proof. Consider the network N formed by

- orienting all edges of $I(G)$ from X to Y and giving these arcs capacity $w^V(G)w^E(G)$;
- creating a source s with an arc of capacity $w^E(e)w^V(G)$ going from s to each vertex in X;
- creating a sink t with an arc of capacity $w^V(v)w^E(G)$ entering t from each vertex in Y.

Define an *s-t separator* of N to be a set $S\subseteq\{s\}\cup X\cup Y\cup\{t\}$ such that $s\in S$ and $t\notin S$. Thus, each s-t separator determines a cut in N. By the definition of N, $I(G)$ has a normalized edge weight assignment if and only if N admits a flow of value $w^V(G)w^E(G)$. Since N has a cut of capacity $w^V(G)w^E(G)$, by the max-flow min-cut theorem N has a flow of value $w^V(G)w^E(G)$ if and only if it has no cut of capacity strictly less than $w^V(G)w^E(G)$. Thus we need only show that G is w-balanced if and only if N has no cut of capacity strictly less than $w^V(G)w^E(G)$.

Suppose that G is w-balanced. Let S be an s-t separator in N which determines a cut of minimum capacity. Note that since the cut determined by S has minimum capacity, for every $x\in X$, $x\in S$ if and only if both of the x's neighbors in Y are in S. Thus, if H is the subgraph of G induced by $S\cap Y$, then $E(H)=S\cap X$. Therefore, the capacity of the cut determined by S is

$$w^V(G)w^E(G) - w^V(G)w^E(H) + w^V(H)w^E(G),$$

which is less than $w^V(G)w^E(G)$ only if $wd(G)<wd(H)$. Since G is w-balanced, the capacity of the cut determined by S is at least $w^V(G)w^E(G)$.

On the other hand, suppose that no cut in N has capacity less than $w^V(G)w^E(G)$. Let H be a induced subgraph of G. Define an s-t separator S in N by setting $S=\{s\}\cup E(H)\cup V(H)$. The capacity of the cut determined by S is

$$w^V(G)w^E(G) - w^V(G)w^E(H) + w^V(H)w^E(G).$$

Since no cut has capacity less than $w^V(G)w^E(G)$, we have $wd(H)\leqslant wd(G)$. This implies that G is w-balanced.

The proof of Theorem 1 shows that deciding whether a given weighted graph G is w-balanced is equivalent to finding a minimum cut in a certain network related to G. Thus, any efficient minimum cut algorithm gives a polynomial algorithm for determining whether a given weighted graph G is w-balanced. In the case where G is not w-balanced, we may use the minimum cut to find an induced subgraph H such that $wd(G)<wd(H)$.

2.2 Forming large w-balanced weighted graphs

Now let's consider applying Theorem 1 to combining w-balanced wegihted graphs to form larger ones.

Let G be a weighted graph with vertex-weight function w^V and edge-weight function w^E. Suppose that $S\subseteq V(G)$, then by $G[S]$ we denote the subgraph induced by S. For each vertex v in $V(G)$, $N_S(v)$ denotes the set of the vertices in S which are adjacent to v. We define the *weighted degree* of v in S by

$$d_S^{w^E}(v) = \sum_{s\in N_S(v)} w^E(vs).$$

A *regular partition* of $V(G)$ is a partition of $V(G)$ into disjoint sets $V_1, V_2, \ldots, V_m$ such that there exist nonnegative real numbers $k_1, k_2, \ldots, k_m$ with $d_{V_i}^{w^E}(v)=k_i w^V(v)$ for all $i(1\leqslant i\leqslant m)$and $v\notin V_i$.

Lemma 1. Let G be a weighted graph such that V_1 and V_2 form a regular partition of $V(G)$. Suppose that $G_1=G[V_1]$ and $G_2=G[V_2]$ are w-balanced, and $wd(G_1)\geqslant wd(G_2)$. Then, the following three statements are equivalent:

1. G is w-balanced;
2. $wd(G_1)\leqslant wd(G)$;
3. $wd(G_1)\leqslant wd(G_2)+k_1$.

Proof. By the definition of w-balanced graphs, Condition 2 follows from Condition 1 immediately. Condition 2 can be written as

$$\frac{w^E(G_1)}{w^V(G_1)}\leqslant\frac{w^E(G_1)+w^E(G_2)+k_1w^V(G_2)}{w^V(G_1)+w^V(G_2)}.$$

So it is obvious that Condition 2 and Condition 3 are equivalent. Thus we need only to prove that Conditon 2 implies Conditon 1.

Suppose that $wd(G_1)\leqslant wd(G)$. Since G_1 and G_2 are w-balanced, we may have fixed normalized edge assignments for $I(G_1)$ and $I(G_2)$. Our aim is to produce a normalized edge weight assignment for $I(G)$. Note that a typical edge of $I(G)$ is of the form (e,v) where $e\in E(G)$, $v\in V(G)$ and v is an end-vertex of e. Consider the following two cases:

Case 1 $e\in E(G_i)$. Assign (e,v) the weight

$$\frac{w^V(G)}{w^V(G_i)}w,$$

where w is the weight of (e,v) in the normalized edge weight assignment for $I(G_i)$.

Case 2 $v \in V_i$ and e joins v to a vertex in V_j where $i \neq j$. Assign (e,v) the weight

$$\frac{w^E(e)}{k_j}\left(w^E(G) - w^E(G_i)\frac{w^V(G)}{w^V(G_i)}\right).$$

Now let's prove that the above edge weight assignment for $I(G)$ is normalized.

It is clear that the edge weights in Case 1 are nonnegative. Condition 2 ensures

$$w^E(G) - w^E(G_i)\frac{w^V(G)}{w^V(G_i)} \geqslant 0,$$

and thus the weights are nonnegative in Case 2.

Suppose that $e=(u,v) \in E(G_i)$ and the weights of (e,u) and (e,v) in the normalized edge weight assignment for $I(G_i)$ are w_1^* and w_2^*, respectively, then $w_1^* + w_2^* = w^E(e)w^V(G_i)$. By Case 1 we know that the sum of the weights of the edges of $I(G)$ incident to e is

$$\frac{w^V(G)}{w^V(G_i)}w_1^* + \frac{w^V(G)}{w^V(G_i)}w_2^* = w^E(e)w^V(G).$$

If $e=(u,v)$ with $u \in V_1$ and $v \in V_2$, then the sum of the weights of the edges of $I(G)$ incident to e is

$$\frac{w^E(e)}{k_1}\left(w^E(G) - w^E(G_2)\frac{w^V(G)}{w^V(G_2)}\right) + \frac{w^E(e)}{k_2}\left(w^E(G) - w^E(G_1)\frac{w^V(G)}{w^V(G_1)}\right) = w^E(e)w^V(G).$$

Let v be a vertex in V_1. Denote the edges in G which are incident to v and with the other end-vertices in V_1 by $e_1, e_2, \ldots, e_s$, and the weights of $(e_1,v), (e_2,v), \ldots, (e_s,v)$ in the normalized edge weight assignment for $I(G_1)$ by $w_1, w_2, \ldots, w_s$, respectively, then $\sum_{i=1}^{s} w_i = w^V(v)w^E(G_1)$. Denote the edges in G which are incident to v and with the other end-vertices in V_2 by $e'_1, e'_2, \ldots, e'_t$, then $\sum_{j=1}^{t} w^E(e'_j) = k_2 w^V(v)$ by the fact that V_1 and V_2 form a regular partition of $V(G)$. Then the sum of the weights of the edges of $I(G)$ incident to v is

$$\frac{w^V(G)}{w^V(G_1)}\sum_{i=1}^{s} w_i + \sum_{j=1}^{t}\frac{w^E(e'_j)}{k_2}\left(w^E(G) - w^E(G_1)\frac{w^V(G)}{w^V(G_1)}\right) = w^V(v)w^E(G).$$

The case $v \in V_2$ can be discussed similarly.

The proof is now complete.

Lemma 2. Let a,b,c,d,x be positive real numbers such that $a/b \leqslant x$ and $c/d \leqslant x$, then $[(a+c)/(b+d)] \leqslant x$.

Theorem 2. Let G be a weighted graph such that $V_1, V_2, \ldots, V_m (m \geqslant 2)$ form a regular partition of $V(G)$ with $wd(G[V_1]) \geqslant wd(G[V_2]) \geqslant \ldots \geqslant wd(G[V_m])$. Suppose that $G[V_1]$, $G[V_2], \ldots, G[V_m]$ are w-balanced and $wd(G[V_i]) \leqslant wd(G[V_{i+1}]) + k_i$ for $1 \leqslant i \leqslant m-1$, then G is w-balanced.

Proof. We proceed by induction on m. The case $m=2$ follows from Lemma 1, so we may assume that $m>2$. By the induction hypothesis, $G[V_1 \cup V_2 \cup \ldots \cup V_{m-2}]$ and $G[V_1 \cup V_2 \cup \ldots$

$\cup V_{m-1}]$ are w-balanced. From the condition of the theorem it is easy to know that $V_1 \cup V_2 \cup \ldots \cup V_{m-2}$ and V_{m-1} form a regular partition of $V_1 \cup V_2 \cup \ldots \cup V_{m-1}$, and $wd(G[V_1 \cup V_2 \cup \ldots \cup V_{m-2}]) \geqslant wd(G[V_{m-2}]) \geqslant wd(G[V_{m-1}])$. By Lemma 1 we have

$$wd(G[V_1 \cup V_2 \cup \ldots \cup V_{m-2}]) \leqslant wd(G[V_{m-1}]) + (k_1 + k_2 + \ldots + k_{m-2}).$$

By the hypothesis of the theorem, we have

$$wd(G[V_1 \cup V_2 \cup \ldots \cup V_{m-1}]) = \frac{w^E(G[V_1 \cup V_2 \cup \ldots \cup V_{m-2}]) + w^E(G[V_{m-1}]) + (k_1 + k_2 + \ldots + k_{m-2})w^V(G[V_{m-1}])}{w^V(G[V_1 \cup V_2 \cup \ldots \cup V_{m-2}]) + w^V(G[V_{m-1}])}.$$

Setting $a = w^E(G[V_1 \cup V_2 \cup \ldots \cup V_{m-2}])$, $b = w^V(G[V_1 \cup V_2 \cup \ldots \cup V_{m-2}])$, $c = w^E(G[V_{m-1}]) + (k_1 + k_2 + \ldots + k_{m-2})w^V(G[V_{m-1}])$ and $d = w^V(G[V_{m-1}])$, it follows from Lemma 2 that

$$\begin{aligned} & wd(G[V_1 \cup V_2 \cup \ldots \cup V_{m-1}]) \leqslant \\ & wd(G[V_{m-1}]) + (k_1 + k_2 + \ldots k_{m-2}) \leqslant \\ & wd(G[V_m]) + (k_1 + k_2 + \ldots k_{m-1}). \end{aligned}$$

From the hypothesis of the theorem, we know that $V_1 \cup V_2 \cup \ldots \cup V_{m-1}$ and V_m form a regular partition of $V(G)$ with $wd(G[V_1 \cup V_2 \cup \ldots \cup V_{m-1}]) \geqslant wd(G[V_{m-1}]) \geqslant wd(G[V_m])$. Note that $G[V_1 \cup V_2 \cup \ldots \cup V_{m-1}]$ and $G[V_m]$ are w-balanced, this implies that G is w-balanced.

Corollary 1. Let G be a weighted graph. Suppose that $V_1, V_2, \ldots, V_m$ form a regular partition of $V(G)$ with $wd(G[V_1]) = wd(G[V_2]) = \ldots = wd(G[V_m])$ and that $G[V_i]$ is w-balanced for $1 \leqslant i \leqslant m$, then G is w-balanced.

Let G be a weighted graph with vertex-weight function w_G^V and edge-weight function w_G^E, and let H be a weighted graph with vertex-weight function w_H^V and edge-weight function w_H^E. G and H are said to be *isomorphic* if there is bijection $\theta: V(G) \to V(H)$ such that $w_G^V(v) = w_H^V(\theta(v))$ and $e = (u, v) \in E(G)$ if and only if $e' = (\theta(u), \theta(v)) \in E(H)$ and $w_G^E(e) = w_H^E(e')$ for any vertices $u, v \in V(G)$.

Theorem 3. Let G be a balanced graph with $V(G) = \{1, 2, \ldots, m\}$ and H a w-balanced graph. Suppose that P is a weighted graph with vertex-weight function w^V and edge-weight function w^E, and k is a positive number such that $V(P)$ can be partitioned into disjoint sets $V_1, V_2, \ldots, V_m$ with $P[V_i]$ isomorphic to H for $1 \leqslant i \leqslant m$, $d_{V_j}^{w^E}(v) = kw^V(v)$ whenever $v \in V_i$ and the edge $(i, j) \in E(G)$, and $d_{V_j}^{w^E}(v) = 0$ whenever $v \in V_i$ and the edge $(i, j) \notin E(G)$. Then P is w-balanced.

Proof. Regard the graph G as a weighted graph such that each vertex and each edge has unit weight. By Theorem 1 we may assume that we have fixed normalized edge weight assignments for $I(G)$ and $I(H)$, and thus for $I(P[V_i])$ with $1 \leqslant i \leqslant m$. Our aim is to produce a normalized edge weight assignment for $I(P)$. Note that a typical edge of $I(P)$ is of the form (e, v), where $e \in E(P)$ and $v \in V(P)$, and v is an end-vertex of e. Consider the following two cases:

Case 1 $e \in E(P[V_i])$ for some i. Assign (e,v) the weight $\nu(G)w$, where w is the weight of (e,v) in the normalized edge weight assignment for $I(P[V_i])$.

Case 2 $v \in V_i$ and e joins v to a vertex in V_j with $i \neq j$. Assign (e,v) the weight $w^E(e)w^V(H)w$, where w is the weight of $((i,j,),i)$in the normalized edge weight assignment for $I(G)$.

Now let's prove that the above edge weight assignment for $I(P)$ is normalized.

By the discussions in Case 1 and Case 2, it is clear that the weights of the edges in $I(P)$ are nonnegative.

Suppose that the $e=(u,v) \in E(P[V_i])$ for some i and the weights of (e,u) and (e,v) in the normalized edge weight assignment for $I(P[V_i])$ are w_1^* and w_2^*, respectively, then $w_1^* + w_2^* = w^E(e)w^V(H)$. By Case 1 the sum of the weights of the edges of $I(P)$ incident to e is

$$\nu(G)w_1^* + \nu(G)w_2^* = w^E(e)w^V(H)\nu(G) = w^E(e)w^V(P).$$

If $e=(u,v)$ with $u \in V_i$ and $v \in V_j$ such that $i \neq j$, denote the weights of $(e,u)$and (e,v) in the normalized edge weight assignment for $I(G)$ by w_1^{**} and w_2^{**}, then $w_1^{**} + w_2^{**} = \nu(G)$. By Case 2 the sum of the weights of the edges of $I(P)$ incident to e is

$$w^E(e)w^V(H)w_1^{**} + w^E(e)w^V(H)w_2^{**} = w^E(e)w^V(H)\nu(G) = w^E(e)w^V(P).$$

Let v be a vertex in V_i. Suppose that the edges in $P[V_i]$ which are incident to v are e_1, $e_2, \ldots, e_s$, and $w_1, w_2, \ldots, w_s$ are the weights of $(e_1,v), (e_2,v), \ldots, (e_s,v)$ in the normalized edge weight assignment for $I(P[V_i])$, respectively, then $\sum_{p=1}^{s} w_p = w^V(v)w^E(H)$. Suppose that the edges of G which are incident to the vertex i are $e'_1, e'_2, \ldots, e'_t$, and the weights of $(e'_1,i), (e'_2,i), \ldots, (e'_t,i)$ are $w'_1, w'_2, \ldots, w'_t$, respectively, then $\sum_{q=1}^{t} w'_q = \varepsilon(G)$. From the condition of the theorem, we know that the sum of the weights of the edges of $I(P)$ incident to v is

$$\sum_{p=1}^{s} \nu(G)w_p + \sum_{q=1}^{t} kw^V(v)w^V(H)w'_q =$$
$$w^V(v)w^E(H)\nu(G) + kw^V(v)w^V(H)\varepsilon(G) = w^V(v)w^E(P).$$

The proof of the theorem is complete.

§ 3 Finding a subgraph of maximum w-density

As we noted in § 2, any efficient minimum cut algorithm gives a polynomial algorithm for determining whether a given weighted graph G is w-balanced. A natural question is that whether there is an efficient algorithm to find a subgraph of maximum w-density in the case where G is not w-balanced. In this section we will show that the answer to this question is positive.

Let G be a weighted graph with vertex-weight function w^V and edge-weight function w^E. Suppose that the vertex set of G is $V(G)=\{v_1,v_2,\ldots,v_\nu\}$. To each vertex v_i of G, we associate a 0-1 vector $X=(x_1,x_2,\ldots,x_\nu)$, where $x_i=1$ means that v_i is a vertex of H. It is clear that we can restrict the maximization to subgraphs induced by subsets of $V(G)$ in computing the global w-density $wm(G)$ of G. So we can formulate the problem of finding $wm(G)$ and the corresponding subgraphs H as

$$wm(G)=\frac{1}{2}\max_{x_j=0,1,X\neq 0}\Big(\sum_{i=1}^{\nu}\sum_{j=1}^{\nu}w^E((v_i,v_j))x_ix_j\Big)\Big/\sum_{j=1}^{\nu}w^V(v_j)x_j,$$

were $w^E((v_i,v_j))$ stands for the weight of the edge (v_i,v_j) when $(v_i,v_j)\in E(G)$, and $w^E((v_i,v_j))=0$ when $(v_i,v_j)\notin E(G)$.

This problem is a particular case of the following 0-1 programming problem:

$$(\mathrm{F})\quad\begin{cases}\max z(X)=(\sum_{S\in A}a_S\prod_{i\in S}x_i)/(\sum_{T\in B}\prod_{j\in T}x_j)=\dfrac{f(x)}{g(x)}\\ \text{s.t. } x_i=0,1, i=1,2,\ldots,\nu, \text{and } X=(x_1,x_2,\ldots,x_\nu)\neq 0,\end{cases}$$

where A,B are subsets of $\{1,2,\ldots,\nu\}$, $a_S\geqslant 0$ for S such that $|S|\geqslant 2$, $b_T\leqslant 0$ for T such that $|T|\geqslant 2$, $g(X)>0$ for any solution $X\neq 0$, and $z(X)\geqslant 0$ for any solution $X\neq 0$.

As indicated in [8] by Picard and Queyranne, problem (F) can be solved by solving a sequence of minimum-cut problems in a related network. For a graph with ν vertices and ε edges, the approach presented in [8] and [9] to find $wm(G)$ and the corresponding subgraph regquires at most $O(\nu^2\varepsilon\log^2\nu)$ operations.

§ 4 w-balanced weighted graphs and SEW(G,w) games

A cooperative game (N,v) consists of a set N of *players*, and a *value function* $v: 2^N\to\mathbf{R}$ where for each subset $S\subseteq N$, $v(S)$ represents the value obtained by the coalition of players of the subset S without assistance of other players, with $v(\varnothing)=0$. Individual income can be represented by a vector $x: N\to\mathbf{R}$. We consider games with side payments. For simplicity, we write $x(S)=\sum_{i\in S}x_i$. A vector x is called an *imputation* if $x(N)=v(N)$, and $x_i\geqslant v(\{i\})$ for each $i\in N$ (individual rationality). The main issue in cooperative games is how to fairly distribute the income collectively earned by the whole group of the players in the game, cooperating with each other. Emphases on different principles lead to different solution concepts. For example, the core is defined on the concept of subgroup rationality. An imputation x is in the core if it satisfies the subgroup rationality condition: $x(S)\geqslant v(S)$ for each subset $S\subseteq N$.

In this section we consider a special kind of cooperative game which was proposed by Deng in [2]. In this game for each subset $S\subseteq N$,

1. if $|S|=1$, $v(S)=0$,

2. if $|S|=2$, the value $v(S)$ is explicitly given,
3. if $|S| \geqslant 3, v(S) = \sum_{\{i,j\}\subseteq S} v(\{i,j\})$.

This game can be related to a graph such that its vertices represent players and edges represent the values for pairs of players. The value for a subset S of players is the sum of edge values in the subgraph induced by the corresponding subset vertices of the graph. In [3], Deng names it as a game of sum of edge weights, and denotes it by SEW(G,w) for a graph $G=(V,E)$ with edge-weight function $w: E \to \mathbf{R}$. There are realistic situations this game may arise. Consider a telephone network between different countries. Each player is a long distance telephone company for a country. Income is generated from lines connecting phone calls between two countries. The companies will have to negotiate how to distribute the total income.

One important problem is SEW(G,w) games is to check whether a given imputation is a member of the core of this game. It is proved in [4] that this problem is NP-complete in general case. When all the weights of the edges of the graph G are nonnegative, one can test in polynomial time whether an imputation is in the core of an SEW$(G.w)$ game. We will show that this result also follows from our results on w-balanced weighted graphs.

Suppose that all the edge weights of the graph G are nonnegative for an SEW(G,w) game, and x is an imputation. For each vertex $i \in V(G)$ we associate weight $w^V(i)=x_i$, and for each edge $(i,j)\in E(G)$ we associate weight $w^E((i,j))=w(\{i,j\})$. Thus we obtain a weighted graph G with vertex-weight function w^V and edge-weight function w^E. It is obvious that x is an imputation if and only if $wd(G)=1$. The imputation x is in the core of the SEW(G,w) game if and only if the weighted graph G is w-balanced. From Theorem 1 we know that there exist efficient polynomial algorithms for determining whether a given weighted graph is w-balanced. So we have

Theorem 4. When all the weights of the edges of the graph G are nonnegative, one can test in polynomial time whether an imputation is in the core of an SEW(G,w) game.

Another important problem in SEW(G,w) games is that whether its core is empty. It is shown in [4] that this problem is NP-complete in general cases and can be solved in polynomial time if the graph G has no negative edges. In fact, by the above discussion we can deduce the following more strong result.

Theorem 5. When all the weights of the edges of the graph G are nonnegative, the core of an SEW(G,w) game is nonempty.

Proof. We prove the theorem by constructing an imputation and showing it is a member of the core. For each player i, define $x_i^* = \frac{1}{2}\sum_{(i,j)\in E(G)} w(\{i,j\})$. For each vertex $i\in V(G)$ we associate weight $w^V(i)=x_i^*$, and for each edge $(i,j)\in E(G)$ we associate wegiht $w^E((i,j))=w(\{i,j\})$. Thus we obtain a weighted graph G with vertex-weight function w^V and edge-wegiht function w^E. It is clear that $wd(G)=1$, so $x^*=(x_1^*, x_2^*, \ldots, x_n^*)$ is an imputation.

Suppose that H is a subgraph of G induced on $S \subseteq V$, then

$$wd(H) = \frac{w^E(H)}{w^V(H)} = \frac{w^E(H)}{\sum_{i\in S} w^V(i)} = \frac{w^E(H)}{\sum_{i\in S} x_i^*} =$$

$$\frac{w^E(H)}{\frac{1}{2}(2w^E(H) + \sum_{i\in S, j\in V\setminus S} w^E((i,j))} \leqslant 1 = wd(G).$$

Therefore, the weighted graph G is w-balanced and the imputation x^* is a member of the core.

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1. Dept. of Appl. Math., Northwestern Polytechnical Univ., Xi'an 710072, China.
2. Center of Combinatorics, Nankai Univ., Tianjin 300071, China.