Recurrent Sequences and Schur Functions

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Abstract

We show that some classical determinants in the theory of symmetric functions can be interpreted in terms of recurrent sequences. Conversely we generalize determinantal expressions of Schur functions, by taking several recurrent sequences having same characteristic polynomial, or by prolonging sequences to negative indices. Finally, we give some recurrent sequences associated to plethysm of symmetric functions, for example with characteristic polynomial having roots the same powers of the roots of the original characteristic polynomial.

Keywords: recurrent sequence, characteristic polynomial, Schur functions

1 Introduction and Notation

A sequence $T = \{T_n\}_{n\geq 0}$ is (linear, homogenous) *recurrent of order* k if there exists constants $c_1, \ldots, c_k$ such that, for $n \geq k$ (we normalized the coefficient of T_n to 1),

$$T_n + c_1 T_{n-1} + \cdots + c_k T_{n-k} = 0.$$

The polynomial $x^k + c_1 x^{k-1} + \cdots + c_k$ is called the *characteristic polynomial* of the recurrent sequence. Factorizing it totally, we can write the characteristic polynomial as $R(x, \mathbb{A}) = \prod_{a\in\mathbb{A}}(x-a)$, where $\mathbb{A}$ is the "alphabet" of the roots. One can now interpret functions of the T_n's as symmetric functions in $\mathbb{A}$. The interplay of the theory of symmetric functions and the theory of recurrent sequences provides us a powerful tool (cf. [3] and [4]).

In this paper, we shall mostly treat the case of determinants whose entries are elements of different recurrent sequences having same characteristic polynomial. First we show that such determinants are proportional to Schur functions. Thus, one can use symmetric functions to manipulate such determinants, and vice versa. As an application, we derive some determinantal formulas in the theory of symmetric functions, some classical and some others new (Theorem 2.2, Corollaries 2.4 and 3.2). Noting that recurrent sequences can be

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extended to negative indices, we then obtain new determinantal expressions of Schur functions, by prolonging the sequence of complete functions to negative indices (Proposition 4.6). Finally, we exhibit recurrent sequences with characteristic polynomials $\prod(x - a_1^{h_1} \cdots a_n^{h_n})$, where product on all permutations $[h_1, \ldots, h_n]$ of a given vector (Theorem 5.1). The coefficients of such characteristic polynomials can be interpreted as plethysm of elementary symmetric functions with monomial functions.

We first need some conventions. We shall follow [4] rather than [5].

Let $\mathbb{A}, \mathbb{B}$ be two alphabets, the complete symmetric function $S^i(\mathbb{A} - \mathbb{B})$, the elementary symmetric function $\Lambda^i(\mathbb{A} - \mathbb{B})$ and the power sum $\psi^i(\mathbb{A})$ are defined by

$$\sum_{i \geq 0} S^i(\mathbb{A} - \mathbb{B}) z^i = \frac{\prod_{b \in \mathbb{B}}(1 - bz)}{\prod_{a \in \mathbb{A}}(1 - az)},$$

$$\sum_{i \geq 0} \Lambda^i(\mathbb{A} - \mathbb{B}) z^i = \frac{\prod_{a \in \mathbb{A}}(1 + az)}{\prod_{b \in \mathbb{B}}(1 + bz)},$$

and

$$\psi^i(\mathbb{A}) = \sum_{a \in \mathbb{A}} a^i.$$

We extend the definition by putting $S^i(\mathbb{A}) = \Lambda^i(\mathbb{A}) = 0$ for $i < 0$. With these conventions,

$$R(x, \mathbb{A}) = \sum_{i=0}^{k} S^i(-\mathbb{A}) x^{k-i} = \sum_{i=0}^{k} (-1)^i \Lambda^i(\mathbb{A}) x^{k-i}.$$

We use the exponential notation m^k to denote the vector $(m, m, \ldots, m) \in \mathbb{Z}^k$. For any $I = (i_1, \ldots, i_n), J = (j_1, \ldots, j_n) \in \mathbb{N}^n$, $S_{J/I}(\mathbb{A})$ denote the *skew Schur functions* on the alphabet $\mathbb{A}$ defined by

$$S_{J/I}(\mathbb{A}) = \left| S^{j_s - i_r + s - r}(\mathbb{A}) \right|_{1 \leq r, s \leq n}.$$

When $I = 0^n$, they are called *Schur functions* and one writes $S_J(\mathbb{A})$ instead of $S_{J/0^n}(\mathbb{A})$.

In the following, we will fix a positive integer k and an alphabet $\mathbb{A}$ of order k.

2 Recurrent Sequences and Schur Functions

Let $T^{(i)} = \{T_n^{(i)}\}_{n \geq 0}$ $(1 \leq i \leq k)$ be k recurrent sequences with the same characteristic polynomial $R(x, \mathbb{A})$. We denote by $M(\mathbb{A})$ the following matrix

$$M(\mathbb{A}) := \begin{bmatrix} T_0^{(1)} & T_1^{(1)} & T_2^{(1)} & \cdots \\ T_0^{(2)} & T_1^{(2)} & T_2^{(2)} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ T_0^{(k)} & T_1^{(k)} & T_2^{(k)} & \cdots \end{bmatrix}.$$

For any $J = (j_1, \ldots, j_k) \in \mathbb{N}^k$, let $M_J(\mathbb{A})$ be the sub-matrix of $M(\mathbb{A})$ taken on columns $j_1 + 1, j_2 + 2, \ldots, j_k + k$.

The following lemma is immediate.

Lemma 2.1 *Let $T^{(i)} = \{T_n^{(i)}\}_{n\geq 0}$ $(1 \leq i \leq k)$ be k recurrent sequences with the same characteristic polynomial $R(x, \mathbb{A})$. Then, $T^{(1)}, T^{(2)}, \ldots, T^{(k)}$ are linearly independent if and only if*

$$\Delta := \det(M_{0^k}(\mathbb{A})) = \begin{vmatrix} T_0^{(1)} & T_1^{(1)} & \cdots & T_{k-1}^{(1)} \\ T_0^{(2)} & T_1^{(2)} & \cdots & T_{k-1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ T_0^{(k)} & T_1^{(k)} & \cdots & T_{k-1}^{(k)} \end{vmatrix} \neq 0.$$

Now, we have

Theorem 2.2 *Let $T^{(i)} = \{T_n^{(i)}\}_{n\geq 0}$ $(1 \leq i \leq k)$ be k recurrent sequences with the same characteristic polynomial $R(x, \mathbb{A})$. For any $J \in \mathbb{N}^k$, we have*

$$\det(M_J(\mathbb{A})) = \begin{vmatrix} T_{j_1}^{(1)} & T_{j_2+1}^{(1)} & \cdots & T_{j_k+(k-1)}^{(1)} \\ T_{j_1}^{(2)} & T_{j_2+1}^{(2)} & \cdots & T_{j_k+(k-1)}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ T_{j_1}^{(k)} & T_{j_2+1}^{(k)} & \cdots & T_{j_k+(k-1)}^{(k)} \end{vmatrix} = S_J(\mathbb{A}) \cdot \Delta.$$

Especially, when $T^{(1)}, T^{(2)}, \ldots, T^{(k)}$ are linearly independent, we have

$$S_J(\mathbb{A}) = \det(M_J(\mathbb{A}))/\Delta.$$

Proof. Let $\mathbb{S}(\mathbb{A})$ be the infinite matrix

$$\mathbb{S}(\mathbb{A}) = \begin{bmatrix} S^0(\mathbb{A}) & S^1(\mathbb{A}) & S^2(\mathbb{A}) & \cdots \\ S^{-1}(\mathbb{A}) & S^0(\mathbb{A}) & S^1(\mathbb{A}) & \cdots \\ S^{-2}(\mathbb{A}) & S^{-1}(\mathbb{A}) & S^0(\mathbb{A}) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

and $\mathbb{S}_J(\mathbb{A})$ be the sub-matrix of $\mathbb{S}(\mathbb{A})$ taken on columns $j_1+1, j_2+2, \ldots, j_k+k$. Consider the product $M(\mathbb{A})\mathbb{S}(-\mathbb{A})\mathbb{S}_J(\mathbb{A})$. Since $T_0^{(i)}, T_1^{(i)}, \ldots$ $(1 \leq i \leq k)$ are recurrent sequences with characteristic polynomial $R(x, \mathbb{A})$, the elements in n-th column of $M(\mathbb{A})\mathbb{S}(-\mathbb{A})$ are all null for $n > k$. Hence,

$$\det\big(M(\mathbb{A})\mathbb{S}(-\mathbb{A})\mathbb{S}_J(\mathbb{A})\big) = \det\big(M(\mathbb{A})\mathbb{S}(-\mathbb{A})\big)_{k\times k} \det\big(\mathbb{S}_J(\mathbb{A})_{k\times k}\big),$$

where $M_{k\times k}$ denote the $k \times k$ sub-matrix of M taken on the first k rows and k columns. Therefore,

$$\begin{aligned}
\det M_J(\mathbb{A}) &= \det(M(\mathbb{A})\mathbb{S}_J(\emptyset)) \\
&= \det\big(M(\mathbb{A})\mathbb{S}(-\mathbb{A})\mathbb{S}_J(\mathbb{A})\big) \\
&= \det\big(M(\mathbb{A})\mathbb{S}(-\mathbb{A})\big)_{k\times k} \cdot \det \mathbb{S}_J(\mathbb{A})_{k\times k} \\
&= \det M(\mathbb{A})_{k\times k} \cdot \det \mathbb{S}(-\mathbb{A})_{k\times k} \cdot \det \mathbb{S}_J(\mathbb{A})_{k\times k} \\
&= \Delta \cdot S_J(\mathbb{A}). \qquad \blacksquare
\end{aligned}$$

As a corollary, we derived the classical definition of Schur functions (see [5, Section 1.3] and [8, Section 7.15]).

Corollary 2.3 *Let $V(\mathbb{A})$ be infinite Vandermonde matrix*

$$V(\mathbb{A}) := \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots \\ 1 & a_2 & a_2^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & a_k & a_k^2 & \cdots \end{bmatrix}$$

and $V_J(\mathbb{A})$ be the determinant of the sub-matrix of $V(\mathbb{A})$ taken on columns $j_1+1, j_2+2, \ldots, j_k+k$. For any $J \in \mathbb{N}^k$, we have

$$V_J(\mathbb{A}) = S_J(\mathbb{A})V_{0^k}(\mathbb{A}).$$

If all elements of $\mathbb{A}$ are distinct, $V_{0^k}(\mathbb{A}) \neq 0$ and hence,

$$S_J(\mathbb{A}) = V_J(\mathbb{A})/V_{0^k}(\mathbb{A}).$$

Proof. Clearly, for any $a \in \mathbb{A}$,

$$a^n + S^1(-\mathbb{A})a^{n-1} + \cdots + S^n(-\mathbb{A}) = S^n(a - \mathbb{A}) = 0, \quad n \geq k.$$

Hence, $1, a, a^2, \ldots$ is a recurrent sequence with characteristic polynomial $R(x, \mathbb{A})$. By Theorem 2.2, we get the result immediately. $\blacksquare$

Corollary 2.4 *Let $T = \{T_n\}_{n\geq 0}$ be a recurrent sequence with characteristic polynomial $R(x, \mathbb{A})$. For any $I, J \in \mathbb{N}^k$, we have*

$$\begin{vmatrix} T_{i_1+j_1} & T_{i_1+j_2+1} & \cdots & T_{i_1+j_k+(k-1)} \\ T_{i_2+j_1+1} & T_{i_2+j_2+2} & \cdots & T_{i_2+j_k+k} \\ \vdots & \vdots & \cdots & \vdots \\ T_{i_k+j_1+(k-1)} & T_{i_k+j_2+k} & \cdots & T_{i_k+j_k+2(k-1)} \end{vmatrix}$$
$$= S_I(\mathbb{A}) \cdot S_J(\mathbb{A}) \cdot \begin{vmatrix} T_0 & T_1 & \cdots & T_{k-1} \\ T_1 & T_2 & \cdots & T_k \\ \vdots & \vdots & \cdots & \vdots \\ T_{k-1} & T_k & \cdots & T_{2(k-1)} \end{vmatrix}.$$

Proof. By Theorem 2.2, we have

$$\begin{aligned}
& S_I(\mathbb{A}) \cdot S_J(\mathbb{A}) \cdot \det(T_{r+s-2})_{1\le r,s\le k} \\
= \; & S_I(\mathbb{A}) \cdot \det(T_{r+(s+j_s)-2})_{1\le r,s\le k} \\
= \; & S_I(\mathbb{A}) \cdot \det(T_{(r+j_r)+s-2})_{1\le r,s\le k} \\
= \; & \det(T_{(r+j_r)+(s+i_s)-2})_{1\le r,s\le k} \\
= \; & \det(T_{i_r+j_s+(r+s-2)})_{1\le r,s\le k}
\end{aligned}$$

∎

Corollary 2.5 *We have*

$$\det(\psi_{i_r+j_s+r+s-2}(\mathbb{A}))_{1\le r,s\le k} = S_I(\mathbb{A})S_J(\mathbb{A})\det(\psi_{r+s-2}(\mathbb{A}))_{1\le r,s\le k}.$$

Proof. Since $1, a, a^2, \ldots$ $(a \in \mathbb{A})$ are recurrent sequences with characteristic polynomial $R(x, \mathbb{A})$, so does their sum $\psi_0(\mathbb{A}), \psi_1(\mathbb{A}), \ldots$. This fact also results from the formulas of Newton relating power sums and elementary symmetric functions, which first appeared in Newton's book *Arithmetica universalis*. (See also [6, 10].) ∎

Notice also that $\det(\psi_{r+s-2}(\mathbb{A}))_{1\le r,s\le k}$ is the discriminant of $\mathbb{A}$, i.e. the square of the Vandermonde $V_{0^k}(\mathbb{A})$.

Corollary 2.6 *Let I, J be two vectors in $\mathbb{N}^k$ and $\square = r^k \subseteq I$, then*

$$S_{I-\square}(\mathbb{A})S_{J+\square}(\mathbb{A}) = S_I(\mathbb{A})S_J(\mathbb{A}),$$

where $J + \square = [j_1 + r, \ldots, j_k + r]$, $I - \square = [i_1 - r, \ldots, i_k - r]$.

Especially,

$$S_{J+\square}(\mathbb{A}) = S_J(\mathbb{A}) \cdot S_\square(\mathbb{A}) = S_J(\mathbb{A}) \cdot (\Lambda^k(\mathbb{A}))^r.$$

Proof. Expanding $S^m(\mathbb{A} - \mathbb{A}) = 0, m > 0$, one sees that $\{S^{-k+1+n}(\mathbb{A})\}_{n\ge 0}$ is a recurrent sequence with characteristic polynomial $R(x, \mathbb{A})$. Taking $T_n = S^{-k+1+n}(\mathbb{A})$ in Corollary 2.4, we reach the conclusion. ∎

Remark. More generally, the factorization lemma given in [4, Prop 9] implies that

$$S_{I-\square}(\mathbb{A})S_{J+\square}(\mathbb{A} - \mathbb{B}) = S_I(\mathbb{A} - \mathbb{B})S_J(\mathbb{A}),$$

and

$$S_{J+\square}(\mathbb{A} - \mathbb{B}) = S_J(\mathbb{A})S_\square(\mathbb{A} - \mathbb{B}),$$

where $\square = m^k$, m being the order of B.

3 Mixed Determinants

Theorem 3.1 *Let X be an arbitrary $m \times (m + k)$ matrix and Y, Z be $k \times (m + k)$ matrices such that each row is a recurrent sequence with characteristic* *polynomial* $R(x,\mathbb{A})$. *Write* $X=[X_1,X_2]$, $Y=[Y_1,Y_2]$ *and* $Z=[Z_1,Z_2]$ *such that* X_2, Y_1, Z_1 *are square matrices. If*

$$\begin{vmatrix} X_1 & X_2 \\ Z_1 & Z_2 \end{vmatrix} \neq 0,$$

we have $\det(Z_1)\neq 0$ *and*

$$\begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix} \Big/ \begin{vmatrix} X_1 & X_2 \\ Z_1 & Z_2 \end{vmatrix} = \det(Y_1)/\det(Z_1)$$

is independent of X.

Proof. Write

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} S^0(-\mathbb{A}) & S^1(-\mathbb{A}) & \cdots & S^{m+k-1}(-\mathbb{A}) \\ 0 & S^0(-\mathbb{A}) & \cdots & S^{m+k-2}(-\mathbb{A}) \\ \dots & \dots & \vdots & \dots \\ 0 & 0 & \cdots & S^0(-\mathbb{A}) \end{bmatrix},$$

where A, B and C are $k\times k, k\times m$ and $m\times m$ matrices respectively.

Noting that each row of Y is a recurrent sequence with characteristic polynomial $R(x,\mathbb{A})$, we have

$$\begin{aligned} \begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix} &= \begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix} \cdot \begin{vmatrix} A & B \\ 0 & C \end{vmatrix} \\ &= \begin{vmatrix} X_1A & X_1B+X_2C \\ Y_1A & 0 \end{vmatrix} \\ &= -\det(X_1B+X_2C)\det(Y_1A). \end{aligned}$$

Similarly,

$$\begin{vmatrix} X_1 & X_2 \\ Z_1 & Z_2 \end{vmatrix} = -\det(X_1B+X_2C)\det(Z_1A).$$

By hypothesis, $\det(X_1B+X_2C)\neq 0$ and $\det(Z_1A)\neq 0$. Hence,

$$\begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix} \Big/ \begin{vmatrix} X_1 & X_2 \\ Z_1 & Z_2 \end{vmatrix} = \det(Y_1)/\det(Z_1).$$ ∎

Combining Theorems 2.2 and 3.1, we have

Corollary 3.2 *Let* $T=\{T_n\}_{n\geq 0}$ *be a recurrent sequence with characteristic polynomial* $R(x,\mathbb{A})$. *Suppose that*

$$W := \begin{vmatrix} x_{1\,1} & x_{1\,2} & \cdots & x_{1\,m+k} \\ \vdots & \vdots & \dots & \vdots \\ x_{m\,1} & x_{m\,2} & \cdots & x_{m\,m+k} \\ T_0 & T_1 & \cdots & T_{m+(k-1)} \\ T_1 & T_2 & \cdots & T_{m+k} \\ \vdots & \vdots & \dots & \vdots \\ T_{k-1} & T_k & \cdots & T_{m+2(k-1)} \end{vmatrix} \neq 0.$$

Then, for any $I \in \mathbb{N}^k$ we have

$$\frac{1}{W} \cdot \begin{vmatrix} x_{1\,1} & x_{1\,2} & \cdots & x_{1\,m+k} \\ \vdots & \vdots & \cdots & \vdots \\ x_{m\,1} & x_{m\,2} & \cdots & x_{m\,m+k} \\ T_{i_1} & T_{i_1+1} & \cdots & T_{i_1+m+(k-1)} \\ T_{i_2+1} & T_{i_2+2} & \cdots & T_{i_2+m+k} \\ \vdots & \vdots & \cdots & \vdots \\ T_{i_k+(k-1)} & T_{i_k+k} & \cdots & T_{i_k+m+2(k-1)} \end{vmatrix} = S_I(\mathbb{A}),$$

and thus, the left-hand side is independent of $x_{11}, x_{12}, \cdots, x_{m\,m+k}$.

Remark. As pointed by Lascoux, Corollary 3.2 can also be derived from the factorization lemma already mentionned. In fact, one can take the rows of the matrix X to be successive powers of $x_1, x_2, .., x_m$, and the recurrent sequence to be $S^n(\mathbb{A} - \mathbb{B})$, with $\mathbb{B}$ of order less than k. Now, subtracting $\mathbb{X} = \{x_1, \ldots, x_m\}$ in the last k columns, one transforms it into ([4, Lemma 8])

$$S_{I+\square}(\mathbb{A} - \mathbb{B} - \mathbb{X}) = S_I(\mathbb{A}) S_\square(\mathbb{A} - \mathbb{B} - \mathbb{X})$$

with $\square = (m + k - 1)^k$, which is a special case of the factorization lemma.

4 Sequences with Negative Indices

In this section, we will assume that all elements of $\mathbb{A}$ are non-zero.

Lemma 4.1 (Wronski) *Let $\mathbb{A}^\vee = \{1/a : a \in \mathbb{A}\}$ and*

$$\beta = (-1)^{k-1} \Lambda^k(\mathbb{A}^\vee) = (-1)^{k-1} / \prod_{a \in \mathbb{A}} a.$$

Then the sequence

$$\mathcal{S}_n = \beta S^{-n-k}(\mathbb{A}^\vee), n < 0 \quad \& \quad \mathcal{S}_n = S^n(\mathbb{A}), n \geq 0,$$

i.e.,

$$\ldots, \beta S^3(\mathbb{A}^\vee), \beta S^2(\mathbb{A}^\vee), \beta S^1(\mathbb{A}^\vee), \beta, \underbrace{0, 0, \ldots, 0}_{k-1}, 1, S^1(\mathbb{A}), S^2(\mathbb{A}), \ldots$$

is a recurrent sequence with characteristic polynomial $R(x, \mathbb{A})$.

This lemma is due to Wronski [9]; see Lascoux [4] for more informations about Wronski and symmetric functions.

For $I, J \in \mathbb{Z}^k$, define

$$\Delta_{I,J} := \begin{vmatrix} \mathcal{S}_{i_k+j_1} & \mathcal{S}_{i_k+j_2+1} & \cdots & \mathcal{S}_{i_k+j_k+k-1} \\ \mathcal{S}_{i_{k-1}+j_1-1} & \mathcal{S}_{i_{k-1}+j_2} & \cdots & \mathcal{S}_{i_{k-1}+j_k+k-2} \\ \vdots & \vdots & \cdots & \vdots \\ \mathcal{S}_{i_1+j_1-(k-1)} & \mathcal{S}_{i_1+j_2-(k-2)} & \cdots & \mathcal{S}_{i_1+j_k} \end{vmatrix}$$

We shall write Δ_J instead of $\Delta_{0^k,J}$.

$\Delta_{I,J}$ can be seen as the extension of Schur functions to negative indices. This extension is needed when interpreting Schur functions as characters of the linear group rather than of the symmetric group.

For $I = (i_1, i_2, \ldots, i_k), J = (j_1, j_2, \ldots, j_k) \in \mathbb{Z}^k$, define

$$-I = (-i_1, \ldots, -i_k), \quad I^\omega = (i_k, i_{k-1}, \ldots, i_1) \quad \text{and} \quad \Delta_{J/I} = \Delta_{-I^\omega,J}.$$

Noting that columns of Δ_J can be seen as elements of recurrent sequences with the same characteristic polynomial $R(x, \mathbb{A})$, by Theorem 2.2, we have

Lemma 4.2 *For $m \in \mathbb{Z}$ and $J \in \mathbb{Z}^k$, $\Delta_{J+m^k} = \Delta_J \cdot (\Lambda^k(\mathbb{A}))^m$.*

Proposition 4.3 *For any $I, J \in \mathbb{N}^k$ and $I \subseteq \Box = r^k$,*

$$\Delta_J = S_J(\mathbb{A}), \quad \Delta_{(J+\Box)/I} = S_{(J+\Box)/I}(\mathbb{A}),$$

and

$$\Delta_{-J} = S_{J^\omega}(\mathbb{A}^\vee).$$

Proof. From the definition of $\mathcal{S}_n$, it's easy to see that

$$\Delta_J = S_J(\mathbb{A}), \quad \Delta_{(J+\Box)/I} = S_{(J+\Box)/I}(\mathbb{A}).$$

Moreover,

$$
\begin{aligned}
S_{J^\omega}(\mathbb{A}^\vee) &= \begin{vmatrix} S^{j_k}(\mathbb{A}^\vee) & S^{j_{k-1}+1}(\mathbb{A}^\vee) & \cdots & S^{j_1+k-1}(\mathbb{A}^\vee) \\ S^{j_k-1}(\mathbb{A}^\vee) & S^{j_{k-1}}(\mathbb{A}^\vee) & \cdots & S^{j_1+k-2}(\mathbb{A}^\vee) \\ \vdots & \vdots & \cdots & \vdots \\ S^{j_k-(k-1)}(\mathbb{A}^\vee) & S^{j_{k-1}-(k-2)}(\mathbb{A}^\vee) & \cdots & S^{j_1}(\mathbb{A}^\vee) \end{vmatrix} \\
&= \begin{vmatrix} S^{j_1}(\mathbb{A}^\vee) & S^{j_2-1}(\mathbb{A}^\vee) & \cdots & S^{j_k-(k-1)}(\mathbb{A}^\vee) \\ S^{j_1+1}(\mathbb{A}^\vee) & S^{j_2}(\mathbb{A}^\vee) & \cdots & S^{j_k-(k-2)}(\mathbb{A}^\vee) \\ \vdots & \vdots & \cdots & \vdots \\ S^{j_1+(k-1)}(\mathbb{A}^\vee) & S^{j_2+(k-2)}(\mathbb{A}^\vee) & \cdots & S^{j_k}(\mathbb{A}^\vee) \end{vmatrix} \\
&= \frac{1}{\beta^k} \begin{vmatrix} \mathcal{S}_{-j_1-k} & \mathcal{S}_{-j_2-k+1} & \cdots & \mathcal{S}_{-j_k-1} \\ \mathcal{S}_{-j_1-k-1} & \mathcal{S}_{-j_2-k} & \cdots & \mathcal{S}_{-j_k-2} \\ \vdots & \vdots & \cdots & \vdots \\ \mathcal{S}_{-j_1-2k+1} & \mathcal{S}_{-j_2-2k+2} & \cdots & \mathcal{S}_{-j_k-k} \end{vmatrix} \\
&= \frac{1}{\beta^k}\Delta_{-J+(-k)^k} \\
&= \frac{1}{\beta^k}(\Lambda^k(\mathbb{A}))^{-k}\Delta_{-J} = \Delta_{-J}.
\end{aligned}
$$

∎

Proposition 4.3 allows us to transform Theorem 2.2 into:

Theorem 4.4 *Let $T^{(i)} = \{T_n^{(i)}\}_{n\in\mathbb{Z}}$ $(1 \le i \le k)$ be k recurrent sequences with the same characteristic polynomial $R(x, \mathbb{A})$. For any $J \in \mathbb{Z}^k$, we have*

$$
\begin{vmatrix} T_{j_1}^{(1)} & T_{j_2+1}^{(1)} & \cdots & T_{j_k+(k-1)}^{(1)} \\ T_{j_1}^{(2)} & T_{j_2+1}^{(2)} & \cdots & T_{j_k+(k-1)}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ T_{j_1}^{(k)} & T_{j_2+1}^{(k)} & \cdots & T_{j_k+(k-1)}^{(k)} \end{vmatrix} = \begin{vmatrix} T_0^{(1)} & T_1^{(1)} & \cdots & T_{(k-1)}^{(1)} \\ T_0^{(2)} & T_1^{(2)} & \cdots & T_{(k-1)}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ T_0^{(k)} & T_1^{(k)} & \cdots & T_{(k-1)}^{(k)} \end{vmatrix} \cdot \Delta_J.
$$

Proof. Taking $M \in \mathbb{N}$ sufficient large such that $J + \square \in \mathbb{N}^k, \square = M^k$, we have

$$\begin{aligned}
\det(T^{(r)}_{j_s+s-1})_{1\le r,s\le k} &= S_{J+\square}(\mathbb{A}) \det(T^{(r)}_{-M+s-1})_{1\le r,s\le k} \\
&= \Delta_{J+\square} \det(T^{(r)}_{-M+s-1})_{1\le r,s\le k} \\
&= S_{\square}(\mathbb{A}) \Delta_J \det(T^{(r)}_{-M+s-1})_{1\le r,s\le k} \\
&= \Delta_J \det(T^{(r)}_{s-1})_{1\le r,s\le k}
\end{aligned}$$

∎

Corollary 2.4 can be transformed into:

Corollary 4.5 *Let $T = \{T_n\}_{n\in\mathbb{Z}}$ be a recurrent sequence with characteristic polynomial $R(x, \mathbb{A})$. For any $I, J \in \mathbb{Z}^k$, we have*

$$\begin{vmatrix}
T_{i_k+j_1} & T_{i_k+j_2+1} & \cdots & T_{i_k+j_k+k-1} \\
T_{i_{k-1}+j_1-1} & T_{i_{k-1}+j_2} & \cdots & T_{i_{k-1}+j_k+k-2} \\
\vdots & \vdots & \cdots & \vdots \\
T_{i_1+j_1-(k-1)} & T_{i_1+j_2-(k-2)} & \cdots & T_{i_1+j_k}
\end{vmatrix}$$

$$= \Delta_I \cdot \Delta_J \cdot \begin{vmatrix}
T_0 & T_1 & \cdots & T_{k-1} \\
T_{-1} & T_0 & \cdots & T_{k-2} \\
\vdots & \vdots & \cdots & \vdots \\
T_{-k+1} & T_{-k+2} & \cdots & T_0
\end{vmatrix}.$$

Taking $T_n = \mathcal{S}_n$ in Corollary 4.5, we have

Proposition 4.6

1. For any $I, J \in \mathbb{Z}^k$, let $\blacksquare = m^k \in \mathbb{Z}^k$, then one has :

$$\Delta_{I,J} = \Delta_I \cdot \Delta_J = \Delta_{J,I} \quad \textit{and} \quad \Delta_{\blacksquare} \cdot \Delta_{-\blacksquare} = 1.$$

2. For any $I, J \in \mathbb{N}^k$,

$$\Delta_{J/I} = S_J(\mathbb{A}) S_I(\mathbb{A}^\vee),$$

Especially, for $I \subseteq \square = r^k$

$$S_{(J+\square)/I}(\mathbb{A}) = S_J(\mathbb{A}) S_{\square/I}(\mathbb{A}).$$

For example, for $k = 3$, the following determinants, corresponding to $J = [-1, 1, 4], [-2, 0, 3], [-3, -1, 2]$, are proportional to the Schur function $S_{0,2,5}(\mathbb{A})$, up to a power of $\beta = \Lambda^3(\mathbb{A}^\vee)$.

$$\begin{vmatrix}
S^4(\mathbb{A}) & S^5(\mathbb{A}) & S^6(\mathbb{A}) \\
1 & S^1(\mathbb{A}) & S^2(\mathbb{A}) \\
\frac{1}{\Lambda^3(\mathbb{A})} & 0 & 0
\end{vmatrix}, \quad
\begin{vmatrix}
S^3(\mathbb{A}) & S^4(\mathbb{A}) & S^5(\mathbb{A}) \\
0 & 1 & S^1(\mathbb{A}) \\
\frac{S^1(\mathbb{A}^\vee)}{\Lambda^3(\mathbb{A})} & \frac{1}{\Lambda^3(\mathbb{A})} & 0
\end{vmatrix}, \quad
\begin{vmatrix}
S^2(\mathbb{A}) & S^3(\mathbb{A}) & S^4(\mathbb{A}) \\
0 & 0 & 1 \\
\frac{S^2(\mathbb{A}^\vee)}{\Lambda^3(\mathbb{A})} & \frac{S^1(\mathbb{A}^\vee)}{\Lambda^3(\mathbb{A})} & \frac{1}{\Lambda^3(\mathbb{A})}
\end{vmatrix}$$

5 Recurrent Sequences and Plethysm

Since the beginning of the theory of polynomials, one has looked at transformations of the type $\prod_{a\in\mathbb{A}}(x-a) \to \prod_{a\in\mathbb{A}}(x-a^2)$, or more generally, $\prod_{a\in\mathbb{A}}(x-a) \to \prod_{a\in\mathbb{A}}(x-a^r)$, for fixed r, and tried to describe the coefficients of the transformed polynomial in terms of those of the original one. Following Littlewood, this transformation is called *plethysm* (with a power sum) [5].

We shall consider here plethysm with a monomial function (more generally, we allow negative exponents).

Let $\mathbb{A} = \{a_1, a_2, \ldots, a_k\}$ be an alphabet of order k, with all $a_i \neq 0$, and let $H \in \mathbb{Z}^k$. Define $\mathbb{A}^H$ to be the alphabet whose letters are all the different monomials obtained by permutation from $a_1^{h_1} \cdots a_k^{h_k}$, i.e.

$$\mathbb{A}^H = \{a_1^{r_1} a_2^{r_2} \cdots a_k^{r_k} : (r_1, r_2, \ldots, r_k) \text{ is a permutation of } H\}$$

Suppose $T^{(i)} = \{T_n^{(i)}\}_{n\in\mathbb{Z}}$ $(1 \leq i \leq k)$ be k recurrent sequences with the same characteristic polynomial $R(x, \mathbb{A})$. For any $I, J \in \mathbb{Z}^k$, denote

$$U_{J/I} = \begin{vmatrix} T^{(1)}_{j_1-i_1} & T^{(1)}_{j_2-i_1+1} & \cdots & T^{(1)}_{j_k-i_1+(k-1)} \\ T^{(2)}_{j_1-i_2-1} & T^{(2)}_{j_2-i_2} & \cdots & T^{(2)}_{j_k-i_2+(k-2)} \\ \cdots & \cdots & \cdots & \cdots \\ T^{(k)}_{j_1-i_k-(k-1)} & T^{(k)}_{j_2-i_k-(k-2)} & \cdots & T^{(k)}_{j_k-i_k} \end{vmatrix}.$$

Now we have

Theorem 5.1 *Let $\mathbb{A}$ be of order k, and I, J, H belong to $\mathbb{Z}^k$. Then the sequence $\{U_{(J+nH)/I}\}_{n\in\mathbb{Z}}$ is a recurrent sequence with characteristic polynomial $R(x, \mathbb{A}^H)$.*

Proof. Firstly, we consider the case that all elements of $\mathbb{A}$ are distinct.

By Theorem 4.4,

$$U_{(J+nH)/I} = \Delta_{J+nH} \cdot \begin{vmatrix} T^{(1)}_{-i_1} & T^{(1)}_{-i_1+1} & \cdots & T^{(1)}_{-i_1+(k-1)} \\ T^{(2)}_{-i_2-1} & T^{(2)}_{-i_2} & \cdots & T^{(2)}_{-i_2+(k-2)} \\ \cdots & \cdots & \cdots & \cdots \\ T^{(k)}_{-i_k-(k-1)} & T^{(k)}_{-i_k-(k-2)} & \cdots & T^{(k)}_{-i_k} \end{vmatrix}.$$

Hence, we only need to prove the assertion holds for Δ_{J+nH}. Similarly to

Corollary 2.3, we have $\Delta_{J+nH} = V_{J+nH}(\mathbb{A})/V_{0^k}(\mathbb{A})$. Now,

$$\begin{aligned}
V_{J+nH}(\mathbb{A}) &= \begin{vmatrix} a_1^{j_1+nh_1} & a_1^{j_2+nh_2+1} & \cdots & a_1^{j_k+nh_k+(k-1)} \\ a_2^{j_1+nh_1} & a_2^{j_2+nh_2+1} & \cdots & a_2^{j_k+nh_k+(k-1)} \\ \cdots & \cdots & \cdots & \cdots \\ a_k^{j_1+nh_1} & a_k^{j_2+nh_2+1} & \cdots & a_k^{j_k+nh_k+(k-1)} \end{vmatrix} \\
&= \sum_{\sigma\in\mathfrak{S}_k} (-1)^{inv(\sigma)} a^{(J+nH+\rho)^\sigma} \\
&= \sum_{\sigma\in\mathfrak{S}_k} (-1)^{inv(\sigma)} a^{(J+\rho)^\sigma} (a^{H^\sigma})^n.
\end{aligned}$$

where a^J denote $a_1^{j_1} a_2^{j_2} \cdots a_k^{j_k}$, $\mathfrak{S}_k$ is the permutation group on $\{1, 2, \ldots, k\}$, $inv(\sigma)$ is the inversion number of permutation σ, and

$$(j_1, j_2, \ldots, j_k)^\sigma = (j_{\sigma(1)}, j_{\sigma(2)}, \ldots, j_{\sigma(k)}), \quad \rho = (0, 1, \ldots, k-1).$$

Noting that $\{(a^{H^\sigma})^n\}_{n\in\mathbb{Z}}$ are recurrent sequences with characteristic polynomial $R(x, \mathbb{A}^H)$, so does their linear combination $V_{J+nH}(\mathbb{A})$.

Since $\Delta_{J/I}$, multiplied by some power of $\Lambda^k(\mathbb{A})$, is a polynomial, the assertion remains true when the elements of $\mathbb{A}$ are not all distinct. ∎

Corollary 5.2 *Suppose $0 < m \le k$, $I, J, H \in \mathbb{Z}^m$, then $\{W_{(J+nH)/I}\}_{n\in\mathbb{Z}}$ is a recurrent sequence with characteristic polynomial $R(x, \mathbb{A}^{H^+})$, where*

$$W_{J/I} = \begin{vmatrix} T^{(1)}_{j_1-i_1} & T^{(1)}_{j_2-i_1+1} & \cdots & T^{(1)}_{j_m-i_1+(m-1)} \\ T^{(2)}_{j_1-i_2-1} & T^{(2)}_{j_2-i_2} & \cdots & T^{(2)}_{j_m-i_2+(m-2)} \\ \cdots & \cdots & \cdots & \cdots \\ T^{(m)}_{j_1-i_m-(m-1)} & T^{(m)}_{j_2-i_m-(m-2)} & \cdots & T^{(m)}_{j_m-i_m} \end{vmatrix}$$

and $H^+ = (0, \ldots, 0, h_1, \ldots, h_m)$ is the embedment of H in $\mathbb{Z}^k$.

Proof. Let I^+, J^+ be the embedment of I, J respectively. Take Theorem 5.1 with I^+, J^+, H^+ and the recurrent sequences being

$$T_n^{(i)} = \begin{cases} \mathcal{S}_{n-k} & 1 \le i \le k-m, \\ T_n^{(i-k+m)} & k-m < i \le k. \end{cases}$$ ∎

Suppose $\{a_n\}_{n\in\mathbb{Z}}$ is a sequence which satisfies a linear recurrence of order k. In [2, 7], it has been shown that for any h and b the subsequences $\{a_{hn+b}\}_{n\in\mathbb{Z}}$ also satisfy a linear recurrence of order k. In fact, it is the case that taking $m = 1$ and $H = h, I = 0, J = b$ in Corollary 5.2.

Taking $H = (1, 1, \ldots, 1) \in \mathbb{Z}^m$ in Corollary 5.2, we derive that $\{W_{(J+n^m)/I}\}_{n\in\mathbb{Z}}$ satisfy a linear recurrence of order $\binom{k}{m}$, which is Proposition 4.3 in [3].

Write $e_r = (\underbrace{0, \ldots, 0}_{r-1}, 1, 0, \ldots, 0) \in \mathbb{N}^k$. For $0 \le \alpha < k$, let

$$J_\alpha = (0, \ldots, 0, \underbrace{1, \ldots, 1}_{\alpha}, -\alpha).$$

By Theorem 5.1, $\{\Delta_{J_\alpha+ne_k}\}_{n\in\mathbb{Z}}$ is a recurrent sequence with characteristic polynomial $R(x, \mathbb{A})$. When $n \ge 0$, $\Delta_{J_\alpha+ne_k}$ is the hook function $S_{1^\alpha, n-\alpha}(\mathbb{A})$.

Furthermore,

$$\Delta_{J_\alpha+ne_k} = (-1)^\alpha \Delta_{ne_{k-\alpha}}, \quad n \in \mathbb{Z}$$

and for $n \ge 0$,

$$\Delta_{ne_{k-\alpha}} = S_{ne_{k-\alpha}}(\mathbb{A}) \quad \text{and} \quad \Delta_{-ne_{k-\alpha}} = S_{ne_{\alpha+1}}(\mathbb{A}^\vee).$$

They are the entries of the n-th power of the companion matrix of the polynomial $R(x, \mathbb{A})$ (see [1, 4]):

$$\begin{bmatrix} \Lambda^1(\mathbb{A}) & -\Lambda^2(\mathbb{A}) & \cdots & (-1)^{k-2}\Lambda^{k-1}(\mathbb{A}) & (-1)^{k-1}\Lambda^k(\mathbb{A}) \\ 1 & 0 & \cdots & 0 & 0 \\ & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

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References

[1] W.Y.C. Chen and J.D. Louck, "The combinatorial power of the companion matrix", Linear Algebra and Its Applications, 232 (1996) 261-278.

[2] F.T. Howard, "Generalizations of a Fibonacci identity", in: Applications of Fibonacci Numbers, **8**, F.T. Howard ed., Kluwer Academic Publishers, 1999.

[3] A. Lascoux, "Suites récurrentes linéaires", Advances in Applied Math., **7** (1986) 228–235.

[4] A. Lascoux, "Symmetric function", Nankai University, December 2001, `http://www.combinatorics.net/teach`

[5] I.G. Macdonald, "Symmetric functions and Hall polynomials", Clarendon press, Oxford, 1979.

[6] D.G. Meed, "Newton's indentities", American Math. Monthly **99**, no. 8 (1992) 749–751.

[7] L. Somer, "Congruence relations for k-th order linear recurrences", The Fibonacci Quarterly **27** (1989) 25–31.

[8] R.P. Stanley, "Enumerative Combinatorics", Wadsworth, Inc., 1986.

[9] H. Wronski, "Philosophie de la Technie algorithmique: Loi suprême et universelle; réforme des mathématiques", Paris 1815–1817.

[10] D. Zeilberger, "A combinatorial proof of Newton's identities", Discrete Math. **49** , no. 3 (1984) pp. 319.