A blossoming algorithm for tree volumes of composite digraphs

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Abstract

A weighted composite digraph is obtained from some weighted digraph by replacing each vertex with a weighted digraph. In this paper, we give a beautiful combinatorial proof of the formula for forest volumes of composite digraphs obtained by Kelmans et al. [DIMACS Technical Report 2000-03, 2000]. Moreover, a generalization of this formula is present.

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1. Introduction

The composite graph $G(H_1, \ldots, H_n)$ is obtained from the graph $G$ on $[n]$ by replacing each vertex $i$ by a graph $H_i$. It is a generalization of multipartite graph, whose complexity was studied by Knuth [8] and Kelmans [6]. The number of spanning trees of $G(H_1, \ldots, H_n)$ was obtained by Pak and Postnikov [9]. These papers generalized the encoding of Prüfer [10]. Recently, Kelmans et al. [7] investigated the tree and forest volumes of weighted digraphs with algebraic methods, and deduced a nice formula for forest volumes of composite weighted digraphs.

To give a combinatorial interpretation of Kelmans–Pak–Postnikov’s formula, we notice that the composite digraph can always be obtained from a series of digraph-substitutions. This fact discloses the essence of a composite digraph. Based on it, we construct a bijection between two sets related to oriented trees. The bijection uses the ideas of Joyal [5] and Goulden and Jackson [3], who independently constructed an elegant encoding for bi-rooted trees. The Joyal encoding was also used by Stanley [12], Eğecioğlu and Remmel [1,2], and

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Guo [4] in their approaches to bipartite or multipartite trees. Although a bijective proof of Kelmans–Pak–Postnikov’s formula may be given by using the Prüfer code, the proof applying the Joyal code is much more simple and straightforward.

We introduce the concept of the oriented tree volume of a weighted digraph, and obtain a formula for oriented tree volumes of weighted composite digraphs, which is equivalent to Kelmans–Pak–Postnikov’s formula. Moreover, a generalized form of this formula is present.

2. Notation and terminology

A directed graph or simply a digraph $G$ is a graph $G$ with each edge $uv$ endowed with a direction from $u$ to $v$ or from $v$ to $u$. The edge set and vertex set of $G$ are denoted by $E(G)$ and $V(G)$, respectively. The edge (or called arc) with direction from $u$ to $v$ is denoted by $(u,v)$, and we call $u$ the initial vertex and $v$ the final vertex. The outdegree of a vertex $v$, denoted $\deg^+_{G}(v)$, is the number of edges of $G$ with initial vertex $v$. Similarly, the indegree of $v$, denoted $\deg^-_{G}(v)$, is the number of edges with final vertex $v$. An oriented tree (or in-tree) with root $v$ is a digraph $T$ with $v$ as one of its vertices, such that there is a unique directed path from any vertex $u$ to $v$. The set of spanning oriented trees of $G$ is denoted by $Sp(G)$.

A weighted digraph $G$ is a digraph $G$, such that each edge $e$ of $G$ is associated with an indeterminate $t_e$ (or an element of a commutative ring).

For each vertex $v$ of a weighted digraph $G$, we associate an indeterminate $x_v$. Let $T$ be a spanning oriented tree of $G$. Define the weight $\omega(T)$ and oriented weight $\bar{\omega}(T)$ of $T$ by

$$\omega(T) := \prod_{v \in V(T)} x_v^{\deg^+(v)-1} \cdot \prod_{e \in E(T)} t_e, \quad (2.1)$$

$$\bar{\omega}(T) := \prod_{v \in V(T)} x_v^{\deg^-(v)} \cdot \prod_{e \in E(T)} t_e, \quad (2.2)$$

where $t_e$ is the weight of the edge $e$. Then define the tree volume $f_G(x,t)$ and oriented tree volume $\bar{f}_G(x,t)$ of $G$ to be the polynomials in the variables $(x_v)_{v \in V(G)}$ and $(t_e)_{e \in E(G)}$ by

$$f_G(x,t) := \sum_{T \in Sp(G)} \omega(T), \quad (2.3)$$

$$\bar{f}_G(x,t) := \sum_{T \in Sp(G)} \bar{\omega}(T). \quad (2.4)$$

Let $G = (V,E)$ be a weighted digraph, $\mathcal{G} = \{G_v: \ v \in V\}$ and $\mathcal{H} = \{H_v: \ v \in V\}$ be two families of disjoint weighted digraphs such that $H_v$ is a subgraph of $G_v$ for every $v \in V$. We construct a new digraph $\Gamma = G(\mathcal{G}, \mathcal{H})$ as follows:
(i) \( V(\Gamma) = \bigcup_{v \in V} V(G_v); \)
(ii) \( E(\Gamma) = \bigcup_{v \in V} E(G_v) \cup \{(x, y) \mid x \in V(H_u), y \in V(H_v), \text{ and } (u, v) \in E(G)\}; \)
(iii) for \( x \in V(H_u), y \in V(H_v), \text{ and } (u, v) \in E(G), \) the edge \((x, y)\) is endowed with the weight \( t(x, y) = t(u, v). \)

We call \( \Gamma \) the composition of \( G \) through \((G, H)\). Denote by \( G(H) = G(G, H) \).

If \( G_u = G_1, H_u = H_1, \text{ and } G_v = H_v = \{v\} \) for \( v \neq u \), then \( G(H) \) is denoted by \( G[G_1, H_1, u] \) and we call \( G[G_1, H_1, u] \) a substitution of \((G_1, H_1)\) into \( G \) in place of \( u \).

Let \( G \) be a weighted digraph, we construct a weighted digraph \( G^* \) as follows:

(i) \( V(G^*) = V(G) \cup *, \) where \( * \notin V(G); \)
(ii) \( E(G^*) = E(G) \cup \{(v, *) \mid v \in V(G)\}; \)
(iii) each edge \((v, *)\) is weighted by \( t(v, *) = 1. \)

We call \( G^* \) the cone of \( G \). For convenience, we also write \( G^* \) for the cone of \( G \) with the new vertex \(*. \)

Now define the forest volume of \( G(H) \) to be the tree volume of the digraph \( G(H)^* \).

**Theorem 2.1** (Kelmans et al. [7, Theorem 11.2]). We have

\[
\bar{f}_{G(H)}^*(x, t) = \bar{f}_{G^*}^*(x, t) |_{x_v = h_v(x)} \prod_{v \in V} \bar{f}_{H_v^*}^*(x_v + g_v(x, t), \quad (2.5)
\]

where \( h_v(x) = \sum_{u \in V(H_v)} x_u \) and \( g_v(x, t) = \sum_{(v, a) \in E(G)} h_a(x) t_{(v, a)}. \)

We will prove the above theorem in the last section.

3. The complete bipartite digraph

Let \( R = \{1, 2, \ldots, r\}, S = \{r + 1, r + 2, \ldots, r + s\}, \) and let \( K_{r+s}^{p,q} \) be the complete bipartite digraph with vertex set partitioned into \( R \cup S, \) and for any \( i \in R \) and \( j \in S, \) the arcs \((i, j)\) and \((j, i)\) are weighted by \( t_{(i, j)} = p_i \) and \( t_{(j, i)} = q_j, \) respectively.

**Lemma 3.1.** We have

\[
\bar{f}_{K_{r+s}^{p,q}}^*(x, t) = p_1 \cdots p_r (q_1 x_1 + \cdots + q_r x_r)^{r-1} (x_{r+1} + \cdots + x_{r+s})^{r-1} \cdot (x_1 q_1 / p_1 + \cdots + x_r q_r / p_r + x_{r+1} + \cdots + x_{r+s}). \quad (3.1)
\]

We will give a bijective proof of Lemma 3.1. The bijection established here is a little different from Eğecioğlu and Remmel [1] and the second solution of Stanley [12, pp. 125–126].

The following definitions and lemma play an important role.
Definition 3.2. An element $a_i$ is called a left-to-right maximum of a permutation $a_1a_2\cdots a_n$ if $a_i > a_j$ for every $j < i$.

Let $\mathcal{S}_n$ denote the set of all permutations on $[n] := \{1, 2, \ldots, n\}$. It is well known that a permutation can be written as a product of its distinct cycles.

Definition 3.3. A standard representation of a permutation is a product of its distinct cycles satisfying that

(a) each cycle is written with its largest element first, and
(b) the cycles are written in increasing order of their largest element.

Definition 3.4. For a permutation $\pi$, we define $\phi(\pi)$ to be the permutation obtained from $\pi$ by writing it in standard form and erasing the parentheses.

Lemma 3.5 (cf. Stanley [11, Proposition 1.3.1]). The map $\phi: \mathcal{S}_n \mapsto \mathcal{S}_n$ defined above is a bijection. If $\pi \in \mathcal{S}_n$ has $k$ cycles, then $\phi(\pi)$ has $k$ left-to-right maxima.

Example 3.6. For $\pi = 24718635$ with standard form $\pi = (412)(6)(73)(85)$, we have $\phi(\pi) = 41267385$ with left-to-right maxima 4, 6, 7, 8.

Note that $\phi^{-1}$ (or $\phi$) is well-defined for any permutation $\pi = a_1a_2\cdots a_k$ over $[n]$ for $k \leq n$. For instance, if $\pi = 517249$, then $\phi^{-1}(\pi) = (51)(724)(9)$.

Proof of Lemma 3.1. Recall that $R = \{1, 2, \ldots, r\}$ and $S = \{1', 2', \ldots, s'\} = \{r + 1, r + 2, \ldots, r + s\}$. We linearly order $R$ and $S$ by $1 < 2 < \cdots < r$ and $1' < 2' < \cdots < s'$, respectively, and we would not compare the elements between $R$ and $S$.

Suppose $T \in \text{Sp}(K^r_{s+t})$ with root $v$. There is a unique directed path $\pi = a_1b_1a_2b_2\cdots a_mb_m$ from $v_0$ to $v$, where $v_0 = 1'$ if $v \in R$, and $v_0 = 1$ if $v \in S$. By Lemma 3.5, we obtain $\phi^{-1}(\pi') = C'_1\cdots C'_k$ from $\pi' = b_1b_2\cdots b_{m-1}$, where $\phi$ is defined in Definition 3.4 and $C'_1, \ldots, C'_k$ are cycles. Then define $C_k$ to be the cycle obtained from $C'_k$ by replacing each $b_i$ with $b_ia_{i+1}$. Specifically, the cycle $C'_k = (b_1b_{i+1}\cdots b_j)$ corresponds to $C_k = (b_ia_{i+1}b_{i+1}a_{i+2}\cdots b_ja_{j+1})$.

Let $D_\pi$ be the disjoint union of the directed cycles $C_1, \ldots, C_k$. When we remove all the edges of the path $\pi$, we obtain a disjoint union of oriented trees. Merge these oriented trees and $D_\pi$ by identifying vertices with the same label. Then we obtain a weighted digraph $\theta_v(T)$ by endowing each arc $e$ with initial (respectively, final) vertex $i$ of $R$ with weight $t_e = p_i$ (respectively, $t_e = q_i$). It is clear that $\theta_v: T \mapsto \theta_v(T)$ is a bijection.

If $v \in R$, then we define the weight of $\theta_v(T)$ by

$$\overline{\omega}(\theta_v(T)) = q_vx_v\prod_{e \in E(\theta_v(T))} t_e \cdot \prod_{i \in R \cup S} x_i^{\deg^{-}(i)}.$$
while if \( v \in S \), then we define the weight of \( \theta_v(T) \) by

\[
\bar{\omega}(\theta_v(T)) = p_1 x_v \prod_{e \in E(\theta_v(T))} t_e \cdot \prod_{i \in R \cup S} \deg^+ (i).
\]

It is easy to see \( \bar{\omega}(T) = \bar{\omega}(\theta_v(T)) \), and \( \bar{\omega}(\theta_v(T)) \) is a term in the expansion of \( p_1 \cdots p_r (q_1 x_1 + \cdots + q_r x_r)^{-1} (x_{r+1} + \cdots + x_{r+s})^{-1} x_v q_v/p_v \) if \( v \in R \), or a term in \( p_1 \cdots p_r (q_1 x_1 + \cdots + q_r x_r)^{-1} (x_{r+1} + \cdots + x_{r+s})^{-1} x_v \) if \( v \in S \).

The lemma follows by summing over all \( v \). \( \square \)

Examples for \( r = 5, s = 8 \) are shown in Fig. 1.

4. The bijections for substituted digraphs

The substituted digraph plays an important part in composite digraphs. We modify the bijection in the argument of Lemma 3.1 to count oriented tree volumes of weighted substituted digraphs. The key idea is to visualize each small tree as a vertex, and a spanning oriented tree of a weighted substituted digraph is then something like a bipartite oriented tree.

Let \( H \) be a subgraph of \( G \). Define \( \mathcal{S}(G, H) \) to be the set of spanning oriented trees of \( G \) with root in \( H \), and denote by

\[
\tilde{f}_{G,H}(x,t) := \sum_{T \in \mathcal{S}(G,H)} \bar{\omega}(T).
\]

**Lemma 4.1.** Let \( G, G_1 \) be disjoint weighted digraphs, \( H_1 \subseteq V(G_1) \), and \( u \in V(G) \). We have

\[
\tilde{f}_{G_1,H_1,u},G(H_1,u)(x,t) = \tilde{f}_{G_1}(x,t)|_{x_u=h_1(t)} \cdot f_{G_1 \cup H_1^*}(x,t)|_{x_u=g_u(t)},
\]

where \( h_1(t) = \sum_{v \in H_1} x_v \) and \( g_u(t) = \sum_{(a,v) \in E(G)} x_v t(a,v) \).

![Fig. 1. The two cases for Lemma 3.1.](image-url)
Proof. Let $G_2 = G \setminus \{u\}$ and $H_2$ be the set of vertices $v$ such that $(u, v) \in E(G)$, and let $\Omega$ be the set of four-tuples $(T_1, T_2, w^{(1)}, w^{(2)})$ such that

(i) $T_1 \in \mathcal{S}p(G_1 \cup H_1^*)$ and $T_2 \in \mathcal{S}p(G)$;
(ii) $w^{(1)}$ and $w^{(2)}$ are words on $H_1$ and $H_2$, respectively;
(iii) $|w^{(1)}| = \deg_{T_2}^-(u)$ and $|w^{(2)}| = \deg_{T_1}^-(*) - 1$.

We want to construct a bijection between $\mathcal{S}p(G[G_1, H_1, u], G[H_1, u])$, and $\Omega$.

First linearly order $H_1$ and $V(G_2)$, respectively.

Suppose $T$ is a spanning oriented tree of $G[G_1, H_1, u]$ with root $r$ in $G[H_1, u]$. Deleting the edges of $T$ between $H_1$ and $G_2$, we obtain weighted oriented forests $F_1$ and $F_2$, which are denoted in $G_1$ and $G_2$, respectively. The oriented tree in $F_1$ or $F_2$ with root $v$ is denoted by $R_v$, and the root sets of $F_1$ and $F_2$ are denoted by $M_1$ and $M_2$, respectively.

If $r \in M_1$, then we define $v_0$ to be $\min M_2$, while if $r \in M_2$, then we define $v_0$ to be $\min M_1$. Assume that $r_1$ is the first vertex on the path from $v_0$ to $r$ in $T$ such that $r_1 \in V(R_r)$.

We may obtain an oriented tree $T_1$ from $F_1$ by adding the vertex $*$, and edges $(v, *)$ ($v \in M_1$), each weighted by 1. To obtain the oriented tree $T_2$ from $F_2$, we add the vertex $u$. If $r \in M_1$, then we add edges $(v, u)$ ($v \in M_2$) with weights in $G$, while if $r \in M_2$, then we add edges $(v, u)$ ($v \in M_2 \setminus \{r\}$) and $(u, r_1)$ with weights in $G$. It remains to find out the words $w^{(1)}$ and $w^{(2)}$.

Identifying each $R_v$ with $v$ in $T$, we obtain a weighted oriented tree $T'$ rooted at $r$. Assume that the directed path from $v_0$ to $r$ in $T'$ is $\pi = p_1 p_2 \cdots p_{2m}$, and $p_1 < p_2 < \cdots < p_{2m}$ are all the left-to-right maxima of $p_2 p_4 \cdots p_{2m-2}$. Let $a_k = p_{i_k-1}$ ($1 \leq k \leq \ell$), and $a_{\ell+1} = p_{2m-1}$.

For any vertex $v \neq v_0$, $r$ of $T'$, let $D(v)$ denote the vertex $z$ such that

\[
\begin{cases}
(a_k, z) \in E(T), & \text{if } v = a_{k+1} (1 \leq k \leq \ell), \\
(v, z) \in E(T), & \text{otherwise}.
\end{cases}
\]

Write $M_1 \setminus \{v_0, r\} = \{u_1, u_2, \ldots, u_q\}$ and $M_2 \setminus \{v_0, r\} = \{v_1, v_2, \ldots, v_q\}$ in increasing order. Put

\[
w^{(1)} = \begin{cases}
D(v_1)D(v_2)\cdots D(v_q)r_1, & \text{if } r \in M_1,
D(v_1)D(v_2)\cdots D(v_q), & \text{if } r \in M_2,
\end{cases}
\]
\[
w^{(2)} = D(u_1)D(u_2)\cdots D(u_q).
\]

It is clear that $|w^{(1)}| = \deg_{T_1}^-(u)$ and $|w^{(2)}| = \deg_{T_1}^-(*) - 1$.

The procedure from $\Omega$ to $\mathcal{S}p(G[G_1, H_1, u], G[H_1, u])$ is as follows:

Given $\tilde{T} = (T_1, T_2, w^{(1)}, w^{(2)}) \in \Omega$. Deleting the vertices $*$ and $u$ of $T_1$ and $T_2$, we get oriented forests $F_1$ and $F_2$, respectively. The oriented tree in $F_1$ or $F_2$ with root $v$ is denoted by $R_v$, and the root sets of $F_1$ and $F_2$ are denoted by $M_1$ and $M_2$, respectively. Suppose the root of $T_2$ is $r'$.
If \( r' = u \), then we define \( v_0 \) to be \( \min M_2, r_1 \) the entry in the last position of \( w(1) \), and \( r \) the root of the oriented tree in \( F \) that contains \( r_1 \). Otherwise, \((u, r_1) \in E(T_2)\) for some vertex \( r_1 \), we denote \( r = r' \), and define \( v_0 \) to be \( \min M_1 \).

Write \( M_1 \setminus \{v_0, r\} = \{u_1, u_2, \ldots , u_q\} \) and \( M_2 \setminus \{v_0, r\} = \{v_1, v_2, \ldots , v_k\} \) in increasing order. For the words \( w(1) \) and \( w(2) \), if we identify each letter (vertex) with the root of the oriented tree in \( F_1 \) or \( F_2 \) that contains it, then we obtain words \( \overline{w}^{(1)} \) and \( \overline{w}^{(2)} \) on \( M_1 \) and \( M_2 \), respectively. By the definition of \( \Omega \), we have \( |\overline{w}^{(1)}| = s + 1 \) or \( s \), according to \( r' = u \) or not, and \( |\overline{w}^{(2)}| = q \). Regard the following function

\[
D_0 = \left( \overline{w}^{(1)}_1, \overline{w}^{(1)}_2, \ldots , \overline{w}^{(2)}_k \right)
\]

as a digraph on \( M_1 \cup M_2 \), such that \( \deg^+(v_0) = \deg^+(r) = 0 \). We may recover the oriented tree \( T' = r^{-1}(D_0) \) with root \( r \) by Lemma 3.1.

Let

\[
D = \left( \overline{w}^{(1)}_1, \overline{w}^{(1)}_2, \ldots , \overline{w}^{(2)}_k \right)
\]

Assume that the path from \( v_0 \) to \( r \) in \( T' \) is \( \pi = p_1 p_2 \cdots p_{2m} \), and \( p_1 < p_2 < \cdots < p_{2m} \) are all the left-to-right maxima of \( p_2 p_3 \cdots p_{2m-2} \). Let \( a_k = p_{2(k-1)} \) \((1 \leq k \leq \ell)\), and \( a_{\ell+1} = p_{2m-1} \). We now connect all the oriented trees in \( F_1 \) or \( F_2 \) by drawing the following edges:

- \( (a_k, D(a_{k+1})) \) \((1 \leq k \leq \ell)\), and \( (p_{2m-1}, r) \);
- \( (v, D(v)) \), for \( v \in V(T') \setminus \{a_1, a_2, \ldots , a_{\ell+1}, r\} \).

Finally, endow the above edges with weights in \( G[G_1, H_1, u] \). Thus we obtain a spanning oriented tree \( T \) of \( G[G_1, H_1, u] \) with root \( r \) (in \( G[H_1, u] \)).

It is not difficult to see that the above two procedures are inverse to each other, therefore we obtain a bijection between \( Sp(G[G_1, H_1, u], G[H_1, u]) \) and \( \Omega \). We now define the weight of \( T = (T_1, T_2, w^{(1)}, w^{(2)}) \) in \( \Omega \) by

\[
\omega(T) = \omega(T_1)\omega(T_2)x_s^{-1 - \deg_T(r)}x_u^-\deg_T(u)\prod_{k \in w^{(1)}} x_k \prod_{v \in w^{(2)}} x_v f_{H_1}(u,v).
\]

It is straightforward to see that the above bijection \( T \mapsto \overline{T} \) is weight-preserving, that is, \( \omega(T) = \omega(\overline{T}) \). Clearly, \( \omega(T) \) is a term of \( f_{G[H_1, u]}(x) \), while \( \omega(\overline{T}) \) is a term in the expansion of \( f_{G}(x) |_{x_u = \sum_{v \in H_1} x_v} \cdot f_{G(H_1,u)}^+(x) |_{x_u = \sum_{v \in H_1} x_v f_{H_1}(u,v)} \). This completes the proof of (4.1). \( \square \)

Examples for Lemma 4.1 are given in Figs. 2–5, where \( H_1 = \{1', 2', \ldots , 14'\} = \{12, 13, \ldots , 25\} \) and \( N = \{1, 2, \ldots , 11\} \) is the set of vertices adjacent to \( u \) in \( G \). We leave out labels of those vertices in \( V(G_1) \setminus H_1 \) or \( V(G_2) \setminus N \). But the root is labeled by \( r \) if necessary.
Fig. 2. Example for Lemma 4.1 in the case \( r \in H_1 \).

Fig. 3. Explanation of \( w^{(1)} \) and \( w^{(2)} \) in Fig. 2.
Fig. 4. Example for Lemma 4.1 in the case \( r \in V(G_2) \).

Fig. 5. Explanation of \( w^{(1)} \) and \( w^{(2)} \) in Fig. 4.
Note that, in the proof of Lemma 4.1, the bijection $T \leftrightarrow (T_1, T_2, w^{(1)}, w^{(2)})$ transforms the root of $T$ into the root of $T_2$ when it belongs to $V(G_2)$, and vice versa. Hence we actually prove the following assertion:

**Lemma 4.2.** Let $G, G_1$ be disjoint weighted digraphs, $G_0$ a subgraph of $G$, $u \in V(G)$, and $H_1 \subseteq V(G_1)$. If $u \in V(G_0)$, then we have

$$\bar{f}_{G(G_1, H_1, u), G(G_0, H_0)}(x, t) = \bar{f}_{G(h), G_0}(x, t) |_{x_u = h_1(x)} \cdot f_{G_1 \cup H_1^*}(x, t) |_{x_v = g_u(x, t)},$$

while if $u \notin V(G_0)$, we have

$$\bar{f}_{G(G_1, H_1, u), G(G_0, H_0)}(x, t) = \bar{f}_{G(h), G_0}(x, t) |_{x_u = h_1(x)} \cdot f_{G_1 \cup H_1^*}(x, t) |_{x_v = g_u(x, t)},$$

where $h_1(x) = \sum_{v \in H_1} x_v$, and $g_u(x, t) = \sum_{(u, v) \in E(G)} x_v t_{(u, v)}$.

5. The blossoming theorem

In the previous section, we obtain a formula for counting the oriented tree volume of a substituted digraph. By using Lemmas 4.1, 4.2, and a series of substitutions of digraphs and variables, we can deduce the main theorem of this paper.

**Theorem 5.1 (The blossoming theorem).** Let $G = (V, E)$ be a weighted digraph, and let $\mathcal{G} = \{G_v : v \in V\}, \mathcal{H} = \{H_v : v \in V\}$ be two families of disjoint weighted digraphs such that $H_v$ is a subgraph of $G_v$ for every $v \in V$. We have

$$\tilde{f}_{G(H), G(H)}(x, t) = \tilde{f}_{G(h), G_0}(x, t) \prod_{v \in V} f_{G_v \cup H_v^*}(x, t) |_{x_v = g_v(x, t)},$$

where $h_v(x) = \sum_{u \in V(H_v)} x_u$ and $g_v(x, t) = \sum_{(v, a) \in E(G)} h_a(x) t_{(v, a)}$. In particular,

$$\tilde{f}_{G(H), G(H)}(x, t) = \tilde{f}_{G(h), G_0}(x, t) \prod_{v \in V} f_{H_v^*}(x, t) |_{x_v = g_v(x, t)}.$$  

**Proof.** Let $V = \{v_1, v_2, \ldots, v_n\}$. We introduce the following notation:

$$G_k := \{G_v : 1 \leq i \leq k\} \cup \{(v_j, \emptyset) : k + 1 \leq j \leq n\},$$

$$H_k := \{H_v : 1 \leq i \leq k\} \cup \{(v_j, \emptyset) : k + 1 \leq j \leq n\},$$

$$h_v^{(k)}(x) := \begin{cases} h_v(x) = \sum_{u \in V(H_v)} x_u, & \text{if } v \in \{v_1, \ldots, v_k\}, \\ x_v, & \text{otherwise}, \end{cases}$$

$$g_v^{(k)}(x, t) := \sum_{(v, a) \in E(G)} h_a^{(k)}(x) t_{(v, a)}, \quad \forall v \in V(G).$$
By the definition of composite digraphs, it is easy to see
\[ G(G_k, H_k) = G(G_{k-1}, H_{k-1})[G_{vk}, H_{vk}, v_k]. \]
\[ G(H_k) = G(H_{k-1})[H_{vk}, v_k]. \]
As we call \((G_{vk}, H_{vk})\) a flower, the process from \(G = G(G_0, H_0)\) to \(G(G_n, H_n)\) is something like blossoming, and any linear ordering of \(V(G)\) leads to the same result \(G = G(G, H)\). Hence, it is proper to call the following proof the blossoming algorithm.

By Lemma 4.2, for \(k \leq m\), we have
\[ \overline{f}_{G(G_k, H_k), G(H_k)}(x, t) = \overline{f}_{G(G_{k-1}, H_{k-1}), G(H_{k-1})}(x, t)|_{x_{vk} = h_{vk}(x), x_{vk} = g(v_k)} \]
It follows that
\[ \overline{f}_{G(G_k, H_k), G(H_k)}(x, t) = \overline{f}_{G(G_{k-1}, H_{k-1}), G(H_{k-1})}(x, t)|_{x_{vk} = h_{vk}(x), x_{vk} = g(v_k)} \]
By iteration of the above equation, we have
\[ \overline{f}_{G(G, H), G(H)}(x, t) = \overline{f}_{G(G_0, H_0), G(H_0)}(x, t)|_{x_{h_a(v)} = h_a(v), 1 \leq a \leq n} \]
This completes the proof of (5.1).

An illustration of the blossoming process is shown in Fig. 6, where the directed connection of \(H_i\) and \(H_j\) (or \(j\)) is meant that every vertex of \(H_i\) is connected to every vertex of \(H_j\) (or the vertex \(j\)) with the same direction and weight as the edge \(ij\).

**Proof of Theorem 2.1.** The proof follows from (5.2) by replacing \(G\) with \(G^\star\) and \(H\) with \(H \cup \{\ast\}\), and the following obvious relations:
\[ \tilde{f}_{G\ast}(x, t)|_{x_{h_a(v)} = g(v)} = \tilde{f}_{G\ast}(x, t)|_{x_{h_a(v)} = g(v)} \]
This completes the proof of (5.1).

\[ \tilde{f}_{G\ast}(x, t)|_{x_{h_a(v)} = g(v)} = \tilde{f}_{G\ast}(x, t)|_{x_{h_a(v)} = g(v)} \]
It is less obvious to see that (2.5) also leads to (5.2) by comparing the terms independent of $x^*$ on both sides of (2.5). Thus they are in fact equivalent to each other.

For a (weighted) graph $G$, $G^*$ is understood to be an undirected graph, and the tree volume $f_G(x, t)$ is defined as (2.3), where $\text{Sp}(G)$ denotes the set of spanning trees of $G$. Eq. (5.2) has many conclusions, we would mention the following:

**Corollary 5.2.** Let $G = (V, E)$ be a weighted graph, and let $\mathcal{H} = \{H_v : v \in V\}$ be a family of disjoint weighted graphs. Then

$$f_G(\mathcal{H})(x, t) = f_G(h, t)|_{h_v = h_v(x)} \times \prod_{v \in V} f_{H_v^*}(x, t)|_{x^* = g_v(x, t)},$$

where $h_v(x) = \sum_{u \in V(H_v)} x_u$ and $g_v(x, t) = \sum_{(v, a) \in E(G)} h_a(x)t(v, a)$.

**Corollary 5.3** (Kelmans [6, Theorem 11]). Let $H_v$ be the empty graph on $v_n$ vertices. Then

$$f_G(H_v)(x) = f_G(h)|_{h_v = h_v(x)} \prod_{v \in V(G)} (g_v(x))^{n_v - 1},$$

where $f_G(x) = f_G(x, t)|_{t_e = 1}$. 

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**Fig. 6.** The blossoming procedure.
Corollary 5.4 (Pak and Postnikov [9]). The number of spanning trees of $G(H_1, \ldots, H_n)$ is equal to

$$\left( \sum_{T \in \text{Sp}(G)} \prod_{v \in V} |H_v|^\deg_T(v)-1 \right) \left( \prod_{v \in V} \sum_{i=1}^{H_v} f_i(H_v) g(v)^{i-1} \right),$$

where $f_i(H_v)$ is the number of forests in $H_v$ with $i$ roots and $g(v) = \sum_{(v,a) \in E} |H_a|$. 

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References