Two Approaches for the Generalization of Leaf Edge Exchange Graphs on Spanning Trees to Connected Spanning k-Edge Subgraphs of a Graph *

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Abstract

In a paper of Harary and Plantholt, they concluded by noting that they knew of no generalization of the leaf edge exchange (LEE) transition sequence result on spanning trees to other natural families of spanning subgraphs. Now, we give two approaches for such a generalization. We define two kinds of LEE-graphs over the set of all connected spanning k-edge subgraphs of a connected graph G, and show that both of them are connected for a 2-connected graph G.

1 Introduction

In [1], Harary and Plantholt investigated the classification of interpolation theorems for spanning trees and other families of spanning subgraphs. They generalized the tree graph $T(G)$ [2] of a graph G to the single edge exchange graph,

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simply called SEE-graph, defined over the set of all connected spanning k-edge subgraphs of G; whereas they generalized the adjacency tree graph $T_a(G)$ [3] of a graph G to the adjacent edge exchange graph, simply called AEE-graph, defined also on the same set of subgraphs of G. They proved that the SEE-graph is connected for any connected graph G; whereas the AEE-graph is connected for any 2-connected graph G. Recently, lower bounds for the connectivity of the SEE-graph and the AEE-graph were obtained by X. Li [4]. However, Harary and Plantholt concluded [1] by noting that they knew of no generalization of the leaf edge exchange transition sequence result on spanning trees to other natural families of spanning subgraphs. In this paper, by viewing leaf edges of a subgraph of a graph G in two different ways, we define two kinds of LEE-graphs, and show that both of them are connected for any 2-connected graph G.

Throughout this paper, all graphs may have multiple or parallel edges but do not have loops. We define a (multi-)graph G to be 2-connected if G has at least 2 vertices and every pair of vertices of G lies in a common cycle of G. So, the definition of the 2-connectedness for a graph is as usual, with only one exception that the graphs with two vertices and multiple edges are 2-connected under our definition. A block of a connected graph G is defined as usual, including the block K_2 with two vertices connected by only one edge. Let H be a subgraph of a graph G. The graph obtained by contracting H from G is the graph that is obtained by identifying all the vertices in H into one vertex, deleting all the loops if there are any, and keeping all the other edges including multiple edges.

2 The First Approach

Definition 2.1 *Let G be a graph and F a subgraph of G. An edge e of F is called a leaf edge of Type 1, or simply T_1-leaf edge whenever $F \setminus \{e\}$ has the same number of components as F does, or one more single vertex component than F does.*

Definition 2.2 *Let $\mathcal{F}$ be the set of all connected spanning k-edge subgraphs of a connected graph G. We define the LEE-graph $T_l^*(G)$ as follows: The vertex set of $T_l^*(G)$ is $\mathcal{F}$; whereas two vertices F and H are adjacent whenever $F \Delta H = \{e, f\}$ such that e and f are T_1-leaf edges of F and H, respectively, where Δ stands for the symmetric difference of two sets.*

In order to show the connectedness of the LEE-graph $T_l^*(G)$, we fit it into the skeleton of the basis graph of a greedoid.

We refer to [5] for terminology and results on greedoids, but repeat the definition for the notions used in this paper.

A greedoid on a finite set E is a pair $(E, \mathcal{I})$, where $\mathcal{I}$ is a nonempty collection of subsets of E satisfying

(1). $\emptyset \in \mathcal{I}$.

(2). For every nonempty $X \in \mathcal{I}$, there is an $x \in X$ such that $X \setminus \{x\} \in \mathcal{I}$.

(3). For $X, Y \in \mathcal{I}$ such that $|X| > |Y|$, there is an $x \in X \setminus Y$ such that $Y \bigcup \{x\} \in \mathcal{I}$.

The sets in $\mathcal{I}$ are called feasible. Thus, a greedoid is a matroid if and only if every subset of a feasible set is again feasible. The maximal feasible sets of a greedoid are called its bases. Like in the case of matroid, (3) implies that all bases of a greedoid have the same cardinality, which is called the rank of the greedoid. Two bases X and Y of a greedoid $(E, \mathcal{I})$ are called adjacent if $|X \bigcap Y| = |X| - 1$ and $X \bigcap Y \in \mathcal{I}$, that is, if X and Y differ in exactly one element and their intersection is feasible. This gives rise to the basis graph $G(\Delta)$ of a greedoid Δ on the set of bases of Δ, whose edges represent pairs of adjacent bases.

A greedoid $(E, \mathcal{I})$ with rank k is called 2-connected if for each $X \in \mathcal{I}$ with $|X| \leq k-2$ there exist $x, y \in E \setminus X$ such that $X \bigcup \{x\}, X \bigcup \{y\}, X \bigcup \{x, y\} \in \mathcal{I}$.

With $G = (V, E, r)$ we denote a finite rooted graph with vertex set V, edge set E, and a specified vertex $r \in V$, which we call the root of G. If X is a subset of E, we denote by $G[X]$ the subgraph of G induced by X.

Lemma 2.1 *Let $G = (V, E, r)$ be a rooted graph, and k a positive integer at least $|V| - 1$. Denote by $\mathcal{I}$ the set of edge sets of all connected subgraphs of G containing the root r such that for $X \in \mathcal{I}$ we have $|X| \leq k$ and $V(G[X]) = V$ if $|X| = k$. Then, $(E, \mathcal{I})$ defines a greedoid, and moreover, the basis graph of $(E, \mathcal{I})$ is a spanning subgraph of the LEE-graph $T_l^*(G)$ over $\mathcal{F}$, defined in Definition 2.2.*

Proof. It is not difficult to verify that the definitions of a greedoid and its basis graph are satisfied by $(E, \mathcal{I})$. The details are omitted. ■

Theorem 2.1 *The LEE-graph $T_l^*(G)$ of a graph G is connected if G is 2-connected.*

Proof. In [6, Theorem 3.1], Korte and Lovász showed that the basis graph of a 2-connected greedoid is connected. Obviously, if G is 2-connected, then the greedoid $(E, \mathcal{I})$ is 2-connected. Therefore, the basis graph $G(E, \mathcal{I})$, and hence the LEE-graph $T_l^*(G)$ by Lemma 2.1, is connected, if G is 2-connected. ■

Notice that in the definition of $T_l^*(G)$, we do not impose the condition that two T_1-leaf edges e and f of F and H, respectively, are adjacent.

Definition 2.3 *We define the adjacent T_1-leaf edge exchange graph, or simply $ALEE$-graph, $T^*_{al}(G)$ as follows: the vertex set is $\mathcal{F}$; whereas two vertices F and H are adjacent whenever $F \Delta H = \{e, f\}$ such that e and f are T_1-leaf edges of F and H, respectively, and e and f share one common vertex.*

Notice that the basis graph of the greedoid $(E, \mathcal{I})$ is no longer a subgraph of $T^*_{al}(G)$. However, the latter is a spanning subgraph of $T^*_l(G)$. As a consequence of Theorem 3.1 in Section 3, we have

Theorem 2.2 *If G is 2-connected, then the $ALEE$-graph $T^*_{al}(G)$ is connected.*

Remark 1. From an algorithmic point of view, the definition of a T_1-leaf edge for a subgraph F of a graph G, in Definition 2.1, is acceptable. This is because the concept of greedoids was introduced for algorithms. For $r = n - 1$, where and in what follows n denotes the number of vertices of G, $(E, \mathcal{I})$ is exactly the undirected branching greedoid [5]. So, a T_1-leaf edge x is exactly an edge of an $X \in \mathcal{I}$ such that $X \setminus \{x\}$ remains inducing a connected subgraph. Thus, for $k \geq n$, $(E, \mathcal{I})$ is indeed a natural generalization of the undirected branching greedoid.

Remark 2. The greedoid $(E, \mathcal{I})$ and its basis graph can be naturally generalized for the directed case. We leave the details to the reader(s).

3 The Second Approach

Definition 3.1 *Let G be a graph, and F a subgraph of G. An edge $e = uv$ of F is called a leaf edge of Type 2, or simply T_2-leaf edge whenever one of the two end-vertices u and v of e is not a cut vertex of F.*

Definition 3.2 *Let $\mathcal{F}$ denote the set of all connected spanning k-edge subgraphs of a connected graph G. We define the LEE-graph $T^{**}_l(G)$ of G as follows: the vertex set is $\mathcal{F}$; whereas two vertices F and H are adjacent whenever $F \Delta H = \{e, f\}$ and e and f are T_2-leaf edges of F and H, respectively.*

Notice that a T_2-leaf edge is a T_1-leaf edge; however, the inverse does not hold. Therefore, $T^{**}_l(G)$ is a spanning subgraph of $T^*_l(G)$.

In a similar way to Definition 2.3, we can give

Definition 3.3 *We define the adjacent T_2-leaf edge exchange graph, or simply $ALEE$-graph, $T_{al}^{**}(G)$ as follows: the vertex set is $\mathcal{F}$, whereas two vertices F and H are adjacent whenever $F\Delta H=\{e,f\}$ such that e and f are T_2-leaf edges of F and H, respectively, and e and f share one common vertex which is not a cut vertex of F and H.*

Obviously, $T_{al}^{**}(G)$ is a spanning subgraph of $T_{al}^{*}(G)$, $T_{l}^{**}(G)$ and $T_{l}^{*}(G)$. The connectedness of $T_{al}^{**}(G)$ and therefore $T_{al}^{*}(G)$, $T_{l}^{**}(G)$ and $T_{l}^{*}(G)$ can be derived from a stronger result which we shall give in the following.

Let $G=(V,E,r)$ be a graph rooted at r. If we never exchange pairs of adjacent T_2-leaf edges with the common vertex r, then we get a restricted adjacent T_2-leaf (or simply, ral) edge exchange graph $T_{ral}^{**}(G)$. In the following we shall show that $T_{ral}^{**}(G)$ is connected if G is 2-connected. Before proceeding, we recall some facts.

Fact 1. If C is a minimal edge cut, then $G\setminus C$ has exactly two (connected) components, or parts.

Fact 2. Let G be 2-connected. If C is a minimal edge cut which separates G into two parts G_1 and G_2, then by contracting any part of G_1 and G_2, the resultant graph G' is still 2-connected.

Fact 3. Let G be a 2-connected graph with at least 3 vertices, and let e, f and g be three pairwise non-parallel edges of G. Then, there is a minimal edge cut of G containing two of e, f and g, say e and f, but not the other one. In fact, since G is 2-connected, any edge cut must contain at least two edges. If $\{e,f\}$ is an edge cut of G, then we are done. If $\{e,f\}$ is not an edge cut, then $G\setminus\{e,f\}$ is connected. Consider a spanning tree T of $G\setminus\{e,f\}$ which contains the edge g. Since G has at least 3 vertices, there is an edge $h\in T\setminus\{g\}$. It is easy to see that there is a minimal edge cut containing e, f and h but not g.

Fact 4. Let G be a 2-connected graph, S a connected spanning subgraph of G. If B is a block of S not isomorphic to K_2, then for any two edges e and f of B there is a minimal edge cut C of G containing e and f such that C separates S into exactly two components. Obviously, any minimal edge cut of S can be extended into a minimal edge cut of G.

Fact 5. Let G and S be as in Fact 4, and C be a minimal edge cut of G that separates S into exactly two components. Suppose the two parts of $G\setminus C$ be G_1 and G_2. Contract any part of G_1 and G_2, say G_2, into one vertex $\bar{r}$. Then, the graph S' contracted from S is a connected spanning subgraph of the graph G' contracted from G. More important, any non-cut-vertex, other than $\bar{r}$, of S' is a non-cut-vertex of S.

Theorem 3.1 *If $G = (V, E, r)$ is a 2-connected graph rooted at r, then the restricted adjacent leaf edge exchange graph $T^{**}_{ral}(G)$ of G is connected.*

Proof. Let F and H be any two vertices of $T^{**}_{ral}(G)$ that represent two connected spanning k-edge subgraphs of G. Since the SEE-graph of G is connected by [1], we can assume that F and H are adjacent in the SEE-graph of G, i.e., $H = F + e \setminus f$, for some two edges e and f of G such that $f \in F$ but $e \notin F$. Since F and $F + e \setminus f$ are connected, we know that both e and f are not cut edges of $F + e$. Thus, e is in a block of $F + e$ which is not isomorphic to K_2, and so is f.

We shall use induction on $|V|$ to complete the proof.

Obviously, G has at least 2 vertices. If $|V| = 2$, then, since G is 2-connected, G is a multi-graph with 2 vertices and at least 2 edges. The conclusion is clear.

Suppose $|V| \geq 3$, and the conclusion holds for any $G = (V, E, r)$ with $|V| < n$.

Assume now $G = (V, E, r)$ with $|V| = n \geq 3$. First we consider the case that e and f are not parallel edges of G. We distinguish the following cases.

Case 1. edges e and f are not parallel, but are in the same block B of $F + e$.

Then, from Fact 4, we consider a minimal edge cut C of G containing e and f such that C separates $F + e$ into exactly two components. Let the two parts of $G \setminus C$ be G_1 and G_2. Without loss of generality, we assume that $r \in G_2$.

Subcase 1.1. $|V(G_2)| \geq 2$.

Then, contract G_2 into one vertex $\bar{r}$. The graph G' contracted from G is 2-connected by Fact 2, and has less than n vertices. By the induction hypothesis, one can transform the graph F' contracted from F into the graph H' contracted from H by exchanges of ral edges in G'. From Fact 5, one can also do the same exchanges of ral edges in G to transform F into H.

Subcase 1.2. $|V(G_2)| = 1$.

Then, $V(G_2) = \{r\}$, i.e., e and f share one common vertex r. Let the other vertex of e and f be u and v, respectively. Since e and f are not parallel, we have $u \neq v$. Since e and f are in a same block B of $F + e$, there is a path P_{uv} in B connecting u and v such that $r \notin V(P_{uv})$. Let h be an edge on P_{uv}. From Fact 3, there are two minimal edge cuts C_1 and C_2 of $F + e$ such that $e, h \in C_1$ but $f \notin C_1$, and $f, h \in C_2$ but $e \notin C_2$.

Step 1. Let the two parts of $G \setminus C_1$ be G_{11} and G_{12} with $r \in G_{12}$. Then, contract G_{12} into one vertex $\bar{r}_1$. Since $|V(G_{12})| \geq 2$, the graph G'_1 contracted from G has less than n vertices. By the induction hypothesis, one can transform the graph

F' contracted from F into the graph $(F+e\setminus h)'$ contracted from $F+e\setminus h$ by exchanges of ral edges in G'_1. From Fact 5, one can also transform F into $F+e\setminus h$ by exchanges of ral edges in G.

Step 2. Let the two parts of $G\setminus C_2$ be G_{21} and G_{22} with $r\in G_{22}$. Then, contract G_{22} into one vertex $\bar{r_2}$. Since $|V(G_{22})|\geq 2$, the graph G'_2 contracted from G has less than n vertices. By the induction hypothesis, one can transform the graph $(F+e\setminus h)'$ contracted from $F+e\setminus h$ into the graph $(F+e\setminus f)'$ contracted from $(F+e\setminus h)+h\setminus f=F+e\setminus f$ by exchanges of ral edges in G'_2. From Fact 5, one can also transform $F+e\setminus h$ into $F+e\setminus f$ by exchanges of ral edges in G.

From Steps 1 and 2, one can successively transform F into $F+e\setminus f$ by exchanges of ral edges in G.

Case 2. edges e and f are not parallel, and are in different blocks of $F+e$ which are not isomorphic to K_2, say $e\in B$ and $f\in B_0$.

Suppose in $F{+}e$ the unique block sequence from B_0 to B is $B_0,B_1,\cdots,B_m=B$, We use induction on the number m to complete the proof.

If $m=1$, then B_0 and B share one common vertex w that is a cut vertex of $F+e$. Consider the components $A_1,\cdots,A_t$ of $(F+e)\setminus\{w\}$. Since G is 2-connected, there is a path P_{fe} in $G\setminus\{w\}$ connecting the edges f and e in G. Let the edges of $E(P_{fe})\setminus(F+e)$ be $e_1,e_2,\cdots,e_p$, in a successive order from f to e. Notice that here we assume that the number of edges in $E(P_{fe})\setminus(F+e)$ is as small as possible. First, consider $F+e_1$. It is easy to see that f and e_1 are in the same block of $F+e_1$. From Case 1 we know that one can transform F into $F+e_1\setminus f$ by exchanges of ral edges in G. Then, consider $(F+e_1\setminus f)+e_2$. It is easy to see that e_1 and e_2 are in a same block of $(F+e_1\setminus f)+e_2$. From Case 1, one can transform $F+e_1\setminus f$ into $(F+e_1\setminus f)+e_2\setminus e_1=F+e_2\setminus f$ by exchanges of ral edges in G. By successively considering $e_3,\cdots,e_p$ and finally e, one can successively transform F into $F+e\setminus f$ by exchanges of ral edges in G.

Now, consider the case $m\geq 2$. Let the shared cut vertex of B_0 and B_1 be w. Consider $(F+e)\setminus\{w\}$. Since G is 2-connected, there is a path P_{fe}in $G\setminus\{w\}$ connecting f and e in G. Suppose the number of edges in $E(P_{fe})\setminus(F+e)$ is as small as possible. Let the edges of $E(P_{fe})\setminus(F+e)$ be $e_1,e_2,\cdots,e_t$, in a successive order from f to e. Then, among the edges e_i for $i=1,2,\cdots,t$, e_t is the unique edge with one end vertex in $\bigcup_{i=1}^{m}B_i$. By the same proof technique as for the case $m=1$, one can transform F into $F+e_t\setminus f$ by exchanges of ral edges in G. Now, consider $F+e_t\setminus f$ and $(F+e_t\setminus f)+e\setminus e_t=F+e\setminus f$. Since in $(F+e_t\setminus f)+e$ the number of blocks from e_t to e is less than m, or e_t and e are in the same block, by induction hypothesis on m, or Case 1, one can transform $F+e_t\setminus f$ into $(F+e_t\setminus f)+e\setminus e_t=F+e\setminus f$ by exchanges of ral

edges in G. Thus, one can successively transform F into $F + e \setminus f$ by exchanges of ral edges in G.

Finally, since G may have multiple edges, we have to consider the case that e and f are parallel edges of G. Let $e = uv$ and $f = uv$. Obviously, e and f are in the same block of $F + e$, say $e, f \in B$.

Case 3. edges e and f are parallel edges uv, in a block B of F_e with $|V(B)| \geq 3$.

Then, there is a vertex $w \in B$, other than u and v, and there is a path P_{uw} in $B \setminus \{v\}$ connecting u and w. Let h be an edge on P_{uw}. Then, e and h are not parallel edges, and so are f and h. Since e and h are in the same block B of $F+e$, from Case 1 one can transform F into $F + e \setminus h$ by exchanges of ral edges in G. Again, since f and h are in the same block B of $(F + e \setminus h) + h = F + e$, from Case 1 one can transform $F + e \setminus h$ into $(F + e \setminus h) + h \setminus f = F + e \setminus f$ by exchanges of ral edges in G. Therefore, one can successively transform F into $F + e \setminus f$ by exchanges of ral edges in G.

Case 4. edges e and f are parallel edges uv, in a block B of $F+e$ with $|V(B)| = 2$.

Then, $V(B) = \{u, v\}$. Since $|V(G)| \geq 3$, one of u and v is a cut vertex of $F + e$, say v. Then, $(F + e) \setminus \{v\}$ has at least two components. Since G is 2-connected, there is an edge h in $E(G) \setminus (F + e)$ such that $h = uw$ for some block $B'(\neq B)$ of $F + e$ with $w \in B'$. Then, h and f are not parallel edges, and so are h and e. It is easy to see that h and f are in a same block of $F + h$. From Case 1, one can transform F into $F + h \setminus f$ by exchanges of ral edges in G. Again, h and e are in a same block of $(F + h \setminus f) + e$. So, one can transform $F + h \setminus f$ into $(F + h \setminus f) + e \setminus h = F + e \setminus f$ by exchanges of ral edges in G. Therefore, one can successively transform F into $F + e \setminus f$ by exchanges of ral edges in G.

Because all possibilities are covered by Cases 1-4, our proof is complete. ■

Corollary 3.1 *If G is 2-connected, then all the four leaf edge exchange graphs $T_l^*(G)$, $T_{al}^*(G)$, $T_l^{**}(G)$ and $T_{al}^{**}(G)$, and the AEE-graph defined in* $[1]$ *are connected.*

Proof. It follows from Theorem 3.1 and the fact that $T_{ral}^{**}(G)$ is a spanning subgraph of all the five graphs in the corollary. ■

By induction on $|V|$, similar to the proof of Theorem 3.1, and the fact that the SEE-graph is connected, we can derive the following corollary, which is similar to a result in [7].

Corollary 3.2 *The diameter of $T^{**}_{ral}(G)$ of a 2-connected graph G is upper bounded by kn^2, where n is the number of vertices of G and k is the number of edges in the considered k-edge subgraphs, representing the vertices of $T^{**}_{ral}(G)$.*

Remark 3. If in Definition 3.3 we only require that e and f are leaf edges of F and H, respectively, and share one common vertex, but drop the condition that the shared vertex is not a cut vertex of F and H, then we get another variation of leaf edge exchange graphs. Since this variation has $T^{**}_{ral}(G)$ as a spanning subgraph, it is connected if G is 2-connected.

Remark 4. One can also obtain the same result of Theorem 3.1 for directed graphs. The details are left to the reader(s).

Remark 5. One can naturally generalize our definitions of leaf edges and their exchanges to other families of subgraphs, mentioned in Corollary $4a$ of [1]. One can also consider applications of our results for interpolations of graph invariants. We leave out the details.

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