ASYMPTOTIC ANALYSIS OF k-NONCROSSING MATCHINGS

EMMA Y. JIN, CHRISTIAN M. REIDYS * AND RITA R. WANG

Abstract. In this paper we study k-noncrossing matchings. A k-noncrossing matching is a labeled graph with vertex set $\{1,\dots,2n\}$ arranged in increasing order in a horizontal line and vertex-degree 1. The n arcs are drawn in the upper halfplane subject to the condition that there exist no k arcs that mutually intersect. We derive: (a) for arbitrary k, an asymptotic approximation of the exponential generating function of k-noncrossing matchings $F_k(z)$. (b) the asymptotic formula for the number of k-noncrossing matchings $f_k(n) \sim c_k\, n^{-((k-1)^2+(k-1)/2)}\,(2(k-1))^{2n}$ for some $c_k>0$.

1. Statement of results and background

Let $F_k(z)$ denote the exponential generating function of k-noncrossing matchings, i.e.

$$F_k(z) = \sum_{n\geq 0} f_k(n)\,\frac{z^{2n}}{(2n)!}\ . \tag{1.1}$$

In this paper we prove the following two theorems:

Theorem 1. *Then we have for arbitrary* $k\in\mathbb{N}$, $k\geq 2$, $\arg(z)\neq\pm\frac{\pi}{2}$

$$F_k(z) \quad = \quad \left[\prod_{i=1}^{k-1}\Gamma(i+1-\frac{1}{2})\prod_{r=1}^{k-2} r!\right]\left(\frac{e^{2z}}{\pi}\right)^{k-1} z^{-(k-1)^2-\frac{k-1}{2}} \quad (1+O(|z|^{-1}))\ . \tag{1.2}$$

Theorem 2. *For arbitrary* $k\in\mathbb{N}$, $k\geq 2$ *we have*

$$f_k(n)\ \sim\ c_k\, n^{-((k-1)^2+(k-1)/2)}\,(2(k-1))^{2n}, \qquad \text{for some } c_k>0\ . \tag{1.3}$$

Date: March, 2008.

Key words and phrases. determinant, Bessel-function, subtraction of singularity principle, oscillating tableaux.

Here we use the notation $f(z) = O(g(z))$ and $f(z) = o(g(z))$ for $|f(z)|/|g(z)|$ being bounded and tending to zero, for $|z| \to \infty$, respectively.

A k-noncrossing matching is a labeled graph over the vertices $1, \dots, 2n$, of degree exactly 1 and drawn in increasing order in a horizontal line. The arcs are drawn in the upper halfplane subject to the condition that there are no k arcs that mutually intersect. Grabiner and Magyar proved an explicit determinant formula, [7] (see also [2], eq. (9)) which expresses the exponential generating function of $f_k(n)$, for fixed k, as a $(k-1) \times (k-1)$ determinant

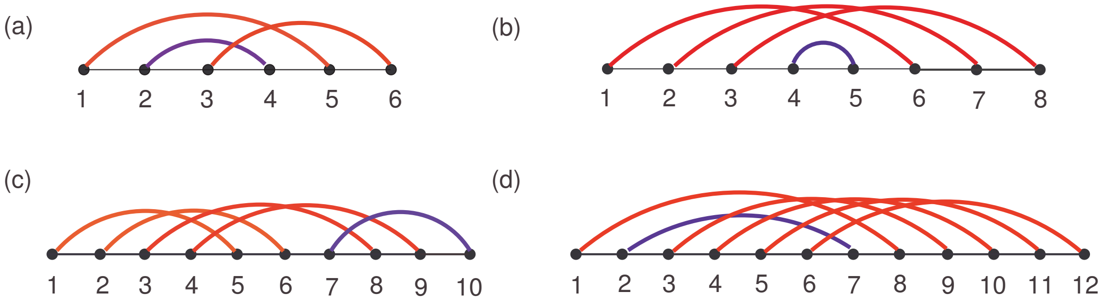

FIGURE 1. k-noncrossing matchings: 3-, 4-, 5- and 6-noncrossing matchings respectively. One of the $k-1$ mutually crossing arcs are drawn in red.

$$F_k(z) = \sum_{n \geq 0} f_k(n) \cdot \frac{z^{2n}}{(2n)!} \quad = \quad \det[I_{i-j}(2z) - I_{i+j}(2z)]|_{i,j=1}^{k-1} , \tag{1.4}$$

where $I_m(2z)$ is the hyperbolic Bessel function:

$$I_m(2z) = \sum_{j=0}^{\infty} \frac{z^{m+2j}}{j!(m+j)!} \quad . \tag{1.5}$$

Chen *et.al.* proved in [2] a beautiful correspondence between k-noncrossing matchings and oscillating tableaux. The particular RSK-insertion used in [2] is based on an idea of Stanley. Our second result is related to a theorem of Regev [10] for the coefficient $u_k(n)$ of Gessel's generating function [5]

$$U_k(x) = \det(I_{i-j}(2z))_{i,j=1}^{k} \ .$$

Regev shows

$$u_k(n) \sim 1!2!\dots(k-1)! \left(\frac{1}{\sqrt{2\pi}}\right)^{k-1} \left(\frac{1}{2}\right)^{(k-1)^2/2} k^{k^2/2} \frac{k^{2n}}{n^{(k^2-1)/2}} \ . \tag{1.6}$$

The proof is obtained employing the RSK-algorithm and using the hook-length formula. One arrives, taking the limit $n \to \infty$, at a k-dimensional *Selberg*-integral, which can be explicitely computed. We shall use a different strategy. One key element in our approach is the following approximation of the Bessel-function, valid for $-\frac{\pi}{2} < \arg(z) < \frac{\pi}{2}$ [1]

$$I_m(z) = \frac{e^z}{\sqrt{2\pi z}} \left(\sum_{h=0}^{H} \frac{(-1)^h}{h! 8^h} \prod_{t=1}^{h} (4m^2 - (2t-1)^2) z^{-h} + \mathrm{O}(|z|^{-H-1}) \right) . \tag{1.7}$$

In this paper we will show that, using the approximation of eq. (1.7), the determinant of Bessel-functions of eq. (1.4) can be computed asymptotically for arbitrary k. The computation of the determinant via the algorithm given in Section 2 is the key ingredient for all our results.

2. Proof of Theorem 1

Suppose we are given a polynomial

$$g_n(x,y) = \sum_{0 \le a+b \le n} C(a,b)\, x^{2a} y^{2b} , \tag{2.1}$$

where for $a+b=n$, $C(a,b) > 0$ holds. In the following a, b always denote integers greater or equal to zero. We set

$$z \triangle z' = (z - z')(z + z') . \tag{2.2}$$

Lemma 1. *Suppose $n \ge 0$, then we have*

$$g_n(x,y) - g_n(x,z) = \begin{cases} (y \triangle z) \sum\limits_{a+b \le n-1} E(a,b,z)\, x^{2a}\, y^{2b} & n \ge 1 \\ 0 & n = 0 \end{cases} \tag{2.3}$$

where $E(a,b,z) = C(a,b+1)$ for $a+b = n-1$. Furthermore

$$g_n(x,y) - g_n(x,y_1) - g_n(x_1,y) + g_n(x_1,y_1) = \begin{cases} (x \triangle x_1)(y \triangle y_1) \sum\limits_{a+b \le n-2} D(a,b,x_1,y_1) x^{2a} y^{2b} & n \ge 2 \\ 0 & n = 0,1 \end{cases} \tag{2.4}$$

where $D(a,b,x_1,y_1) = C(a+1,b+1)$ for $a+b = n-2$.

Proof. For $n=0$, we immediately obtain $g_0(x,y)-g_0(x,z)=0$. In case of $n\geq 1$ we compute

$$\begin{aligned} g_n(x,y)-g_n(x,z) &= (y\triangle z)\sum_{a+b\leq n,\, b>0} C(a,b)\, x^{2a}\left[\sum_{m=0}^{b-1} y^{2m}z^{2b-2-2m}\right] \\ &= (y\triangle z)\sum_{a+b\leq n-1} E(a,b,z)\, x^{2a}y^{2b}. \end{aligned}$$

In particular, for $a+b=n-1$, we observe $E(a,b,z)=C(a,b+1)$. As for eq. (2.4) we compute in case of $n=0$ or 1, $\vartheta_n(x,x_1,y,y_1)=g_n(x,y)-g_n(x,y_1)-g_n(x_1,y)+g_n(x_1,y_1)=0$. For $n\geq 2$ we compute

$$\begin{aligned} \vartheta_n(x,x_1,y,y_1) &= (x\triangle x_1)(y\triangle y_1)\sum_{a+b\leq n,\, ab>0} C_h(a,b)\left[\sum_{m=0}^{a-1} x^{2m}x_1^{2a-2-2m}\right]\left[\sum_{m=0}^{b-1} y^{2m}y_1^{2b-2-2m}\right] \\ &= (x\triangle x_1)(y\triangle y_1)\sum_{a+b\leq n-2} D(a,b,x_1,y_1)\, x^{2a}y^{2b}\ . \end{aligned}$$

In particular, for $a+b=n-2$, $D(a,b,x_1,y_1)=C(a+1,b+1)$ holds. □

Let

$$e_{i,j}(z) = \sum_{h\geq 0} m_h(i,j)\frac{(-1)^h}{16^h h!}\, z^{-h} \tag{2.5}$$

$$m_h(i,j) = \prod_{t=1}^{h}(4(i-j)^2-(2t-1)^2)-\prod_{t=1}^{h}(4(i+j)^2-(2t-1)^2)\ . \tag{2.6}$$

We consider the algorithm **A**, specified in Figure 2, which manipulates the matrix of Laurent series $M=(e_{i,j}(z))_{1\leq i,j\leq k-1}$. Let $e^t_{i,j}(z)$ denote the matrix coefficient after running **A** exactly t steps. We set

$$e^t_{i,j}(z)=\sum_{h\geq 0} m^t_h(i,j)\frac{(-1)^h}{16^h h!}\, z^{-h} \tag{2.7}$$

and proceed by analyzing the terms $m^t_h(i,j)$ for $1\leq t<k-1$.

Lemma 2. *For any positive integer t strictly smaller than $k-1$ and we have $m^t_h(i,j)=m^t_h(j,i)$ the following two assertions hold.*

The algorithm A:

```
begin
$M := [e_{i,j}(z)]_{i,j=1}^{k-1}$;
    for $t$ from 1 to $k-1$ do
        for $i$ from $t+1$ to $k-1$ do
            for $j$ from 1 to $k-1$ do
            $e'_{i,j}(z) := \frac{-i\prod_{r=1}^{t-1}(i\triangle r)}{(2t-1)!}e_{t,j}(z)+e_{i,j}(z)$;
            $e_{i,j}(z) := e'_{i,j}(z)$;
            end;
        end;
        for $j$ from $t+1$ to $k-1$ do
            for $i$ from 1 to $k-1$ do
            $e''_{i,j}(z) := \frac{-j\prod_{r=1}^{t-1}(j\triangle r)}{(2t-1)!}e_{i,t}(z)+e_{i,j}(z)$;
            $e_{i,j}(z) := e''_{i,j}(z)$;
            end;
        end;
    end;
output $M$;
end;
```

FIGURE 2.

(a) *for* $i \le t < j$, *we have*

$$m_h^t(i,j) = -(2i-1)!j\prod_{r=1}^{t}(j\triangle r)\sum_{a+b\le h-(t+i)} E_h^t(a,b,i)\,i^{2a}j^{2b} \tag{2.8}$$

$$h < t+i \implies m_h^t(i,j)=0 \tag{2.9}$$

$$a+b=h-(t+i) \implies E_h^t(a,b,i)=C_h(a+i-1,b+t)>0\ . \tag{2.10}$$

Furthermore (a) *implies the case* $j \le t < i$.

(b) *for $i, j > t$, we have*

$$m_h^t(i,j) = -i\,j \prod_{r=1}^{t}(i\triangle r)\prod_{r=1}^{t}(j\triangle r)\sum_{a+b\le h-(2t+1)} D_h^t(a,b)\,i^{2a}j^{2b} \tag{2.11}$$

$$h < 2t+1 \implies m_h^t(i,j) = 0 \tag{2.12}$$

$$a+b = h-(2t+1) \implies D_h^t(a,b) = C_h(a+t,b+t) > 0\ . \tag{2.13}$$

Proof. We shall prove (a) and (b) by induction on $1 \le t < k-1$. We first observe that, in view of eq. (2.6)

$$\begin{aligned} m_h(i,j) &= -2\sum_{s=0}^{\lfloor\frac{h-1}{2}\rfloor}\sum_{p+q+r+2s+1=h}\binom{h-r}{p}\binom{h-p-r}{q}(4i^2)^p(4j^2)^q a_1\cdots a_r(8ij)^{2s+1} \\ &= -ij\sum_{a+b\le h-1} C_h(a,b)i^{2a}j^{2b}\ , \end{aligned} \tag{2.14}$$

where $a_i \in \{-1^2,\dots,-(2h-1)^2\}$, $i \neq j,\ a_i \neq a_j$ and $C_h(a,b) > 0$ for $a+b = h-1$. Furthermore by definition $m_h(i,j) = m_h(j,i)$. For $i = 1, j > 1$, only the j-loop is executed, whence

$$m_h^1(i,j) = m_h(i,j) - jm_h(i,1) \tag{2.15}$$

and for $m_h(j,i)$ only the i-loop contributes

$$m_h^1(j,i) = m_h(j,i) - jm_h(1,i) = m_h(i,j) - jm_h(i,1) = m_h^1(i,j)\ .$$

Consequently

$$m_h^1(i,j) = -ij\left[\sum_{a+b\le h-1} C_h(a,b)i^{2a}j^{2b} - \sum_{a+b\le h-1} C_h(a,b)i^{2a}\right]$$

Employing Lemma 1 we obtain

$$m_h^1(i,j) = -i\,j\,(j\triangle 1)\sum_{a+b\le h-2} E_h^1(a,b,1)i^{2a}j^{2b}\ .$$

Furthermore

$$\begin{aligned} a+b = h-2 &\implies E_h^1(a,b,1) = C_h(a,b+1) > 0 \\ h = 1 &\implies m_h^1(i,j) = 0\ . \end{aligned}$$

Thus for $t = 1$, the induction basis for (a) holds. We proceed by proving that for $t = 1$ (b) holds. For $i > 1,\ j > 1$ both i- and j-loop are executed

$$m_h^1(i,j) = m_h(i,j) - im_h(1,j) - jm_h(i,1) + ijm_h(1,1) \tag{2.16}$$

from which immediately $m_h^1(i,j) = m_h^1(j,i)$ follows. We compute

$$\begin{aligned} m_h^1(i,j) &= -ij\left(\sum_{a+b\le h-1} C_h(a,b) i^{2a} j^{2b} - \sum_{a+b\le h-1} C_h(a,b) j^{2b}\right) \\ &\quad -ij\left(-\sum_{a+b\le h-1} C_h(a,b) i^{2a} + \sum_{a+b\le h-1} C_h(a,b)\right) \\ &= -i(i\triangle 1)j(j\triangle 1)\sum_{a+b\le h-3} D_h^1(a,b) i^{2a} j^{2b}\ . \end{aligned}$$

Lemma 1 implies for $a+b=h-3$, $D_h^1(a,b) = C_h(a+1,b+1)$ and for $h<3$, $m_h^1(i,j) = 0$. Accordingly we eststablished the induction basis for assertions (a), (b) and $m_h^1(i,j) = m_h^1(j,i)$. As for the induction step, we first prove (a). Let us suppose (a) holds for $t=n$. We consider the case $t=n+1$ by distinguishing subsequent two cases: (1) $i=n+1, j>n+1$ and (2) $i\le n, j=n+1$. First we observe that since $i<n+2$ the algorithm executes no i-loop and by construction the only contribution to $m_h^{n+1}(i,j)$ is made by the term

$$-\frac{j\prod_{r=1}^{n+1-1}(j\triangle r)}{(2n+1)!}\, m_h^n(i,n+1)$$

We accordingly derive

$$m_h^{n+1}(i,j) \quad = \quad m_h^n(i,j) - \frac{j\prod_{r=1}^{n}(j\triangle r)}{(2n+1)!} m_h^n(i,n+1)$$

The induction hypothesis on $t=n$ shows $m_h^n(i,j) = m_h^n(j,i)$ and $m_h^n(i,n+1) = m_h^n(n+1,i)$. Therefore we arrive at $m_h^{n+1}(i,j) = m_h^{n+1}(j,i)$.

(1) $i=n+1, j>n+1$, the induction hypothesis guarantees

$$\begin{aligned} m_h^n(i,j) &= -i\,j\prod_{r=1}^{n}(i\triangle r)\prod_{r=1}^{n}(j\triangle r)\sum_{a+b\le h-(2n+1)} D_h^n(a,b)\, i^{2a} j^{2b} \\ m_h^n(i,n+1) &= -i\,(n+1)\prod_{r=1}^{n}(i\triangle r)\prod_{r=1}^{n}((n+1)\triangle r)\sum_{a+b\le h-(2n+1)} D_h^n(a,b)\, i^{2a}(n+1)^{2b} \end{aligned}$$

Since $(n+1)\prod_{r=1}^{n}((n+1)\triangle r)=(2n+1)!$ we arrive at

$$\begin{aligned} m_h^{n+1}(i,j) &= -i\,j\,\prod_{r=1}^{n}(i\triangle r)\prod_{r=1}^{n}(j\triangle r)\times \\ &\left(\sum_{a+b\leq h-(2n+1)} D_h^n(a,b)i^{2a}j^{2b}-\sum_{a+b\leq h-(2n+1)} D_h^n(a,b)i^{2a}(n+1)^{2b}\right)\ . \end{aligned}$$

According to Lemma 1, for $h=2n+1$, we have $m_h^{n+1}(n+1,j)=0$ and for $h\geq 2n+2$ we obtain

$$m_h^{n+1}(n+1,j) \;=\; -(2n+1)!j\prod_{r=1}^{n+1}(j\triangle r)\sum_{a+b\leq h-(2n+2)} E_h^{n+1}(a,b,n+1)(n+1)^{2a}j^{2b}. \tag{2.17}$$

For $a+b=h-(2n+2)$, we have

$$E_h^{n+1}(a,b,n+1)=D_h^n(a,b+1)=C_h(a+n,b+n+1)>0\ . \tag{2.18}$$

(2) $i\leq n$ and $j>n+1$, using the induction hypothesis, we derive

$$\begin{aligned} m_h^{n+1}(i,j) &= m_h^n(i,j)-\frac{j\prod_{r=1}^{n}(j\triangle r)}{(2n+1)!}m_h^n(i,n+1) \\ &= -(2i-1)!\,j\,\prod_{r=1}^{n}(j\triangle r)\ \times \\ &\left(\sum_{a+b\leq h-(n+i)} E_h^n(a,b,i)i^{2a}j^{2b}-\sum_{a+b\leq h-(n+i)} E_h^n(a,b,i)i^{2a}(n+1)^{2b}\right)\ . \end{aligned}$$

Lemma 1 implies

$$m_h^{n+1}(i,j) \;=\; -(2i-1)!\,j\,\prod_{r=1}^{n+1}(j\triangle r)\sum_{a+b\leq h-(n+1+i)} E_h^{n+1}(a,b,i)i^{2a}j^{2b}.$$

For $h\leq n+i$, we observe $m_h^{n+1}(i,j)=0$ and for $a+b=h-(n+1+i)$,

$$E_h^{n+1}(a,b,i)=E_h^n(a,b+1,i)=C_h(a+i-1,b+n+1)>0\ . \tag{2.19}$$

Accordingly, we have proved

$$\forall\, i\leq n+1<j;\quad m_h^{n+1}(i,j)=-(2i-1)!j\prod_{r=1}^{n+1}(j\triangle r)\sum_{a+b\leq h-(n+1+i)} E_h^{n+1}(a,b,i)i^{2a}j^{2b}\ . \tag{2.20}$$

Furthermore

$$a+b=h-(n+1+i) \implies E_h^{n+1}(a,b,i)=C_h(a+i-1,b+n+1)>0 \tag{2.21}$$

$$h<n+1+i \implies m_h^{n+1}(i,j)=0\ . \tag{2.22}$$

Whence assertion (a) holds by induction for any $1 \le t < k-1$. We next suppose assertion (b) is true for $t=n$ and consider the case $t=n+1$, i.e., $i>n+1$ and $j>n+1$. First the i-loop is executed and produces

$$\tilde{m}_h^{n+1}(i,j)=m_h^n(i,j)-\frac{i\prod_{r=1}^n(i\triangle r)}{(2n+1)!}m_h^n(n+1,j)\ .$$

Secondly the j-loop yields

$$m_h^{n+1}(i,j) \quad = \quad \tilde{m}_h^{n+1}(i,j)-\frac{j\prod_{r=1}^n(j\triangle r)}{(2n+1)!}\tilde{m}_h^{n+1}(i,n+1)\ . \tag{2.23}$$

We accordingly compute

$$\begin{aligned} m_h^{n+1}(i,j) &= m_h^n(i,j)-\frac{i\prod_{r=1}^n(i\triangle r)}{(2n+1)!}m_h^n(n+1,j) \\ &\quad -\frac{j\prod_{r=1}^n(j\triangle r)}{(2n+1)!}\left(m_h^n(i,n+1)-\frac{i\prod_{r=1}^n(i\triangle r)}{(2n+1)!}m_h^n(n+1,n+1)\right) \end{aligned}$$

from which we immediately observe that $m_h^{n+1}(i,j)=m_h^{n+1}(j,i)$ holds. Furthermore

$$\begin{aligned} m_h^{n+1}(i,j) &= -i\left[\prod_{r=1}^n(i\triangle r)\right]j\left[\prod_{r=1}^n(j\triangle r)\right]\times \\ &\quad \sum_{a+b\le h-(2n+1)} D_h^n(a,b)(i^{2a}j^{2b}-(n+1)^{2a}j^{2b}-i^{2a}(n+1)^{2b}+(n+1)^{2a}(n+1)^{2b}) \\ &= -i\left[\prod_{r=1}^{n+1}(i\triangle r)\right]j\left[\prod_{r=1}^{n+1}(j\triangle r)\right]\sum_{a+b\le h-(2(n+1)+1)} D_h^{n+1}(a,b)i^{2a}j^{2b}. \end{aligned}$$

In particular,

$$\begin{aligned} a+b=h-(2(n+1)+1) &\Rightarrow D_h^{n+1}(a,b)=D_h^n(a+1,b+1)=C_h(a+n+1,b+n+1)>0 \\ h=2n+1 \text{ or } h=2n+2 &\Rightarrow m_h^{n+1}(i,j)=0\ . \end{aligned}$$

Thus $m_h^{n+1}(i,j)$ satisfies (b) for any $1 \le t < k-1$. □

We proceed by analyzing the Laurent series

$$a_{i,j}(z) = \sum_{h\ge 0} m_h^{k-2}(i,j)\frac{(-1)^h}{16^h h!}\, z^{-h} \ . \tag{2.24}$$

Lemma 3.

$$a_{i,j}(z) \ = \ (-1)^{i+j}\,\frac{2\Gamma(j+i-\frac{1}{2})}{\sqrt{\pi}} z^{-(j+i-1)}(1+O(|z|^{-1})) \ . \tag{2.25}$$

Proof. We shall prove the lemma distinguishing the cases $i<j$ and $i=j$. The former implies by symmetry the case $i>j$. Suppose first $i<j$. By construction of $\mathbf{A}$, we have

$$m_h^{k-2}(i,j) = m_h^{j-1}(i,j) \tag{2.26}$$

since after the $(j-1)$th step, $m_h^{j-1}(i,j)$ remains unchanged. Consequently we can write $a_{i,j}(z)$ as

$$a_{i,j}(z) = \sum_{0\le h\le i+j-1} \frac{(-1)^h}{16^h\, h!}\, m_h^{j-1}(i,j)\, z^{-h} + \sum_{i+j-1<h} \frac{(-1)^h}{16^h\, h!}\, m_h^{j-1}(i,j)\, z^{-h} \ . \tag{2.27}$$

We consider the terms $m_h^{j-1}(i,j)$ for $0\le h\le j+i-1$. According to Lemma 2

$$m_h^{j-1}(i,j) = -(2i-1)!\, j\prod_{r=1}^{j-1}(j\triangle r) \sum_{0\le a+b\le h-(j-1+i)} E_h^{j-1}(a,b,i)\, i^{2a} j^{2b}$$

holds. In particular,

$$h<j-1+i \quad \Longrightarrow \quad m_h^{j-1}(i,j)=0 \ . \tag{2.28}$$

Accordingly, the only nonzero coefficient of $\sum_{0\le h\le i+j-1}\frac{(-1)^h}{16^h\, h!}\, m_h^{j-1}(i,j)\, z^{-h}$ has index $h=j-1+i$ in which case

$$a+b=0 \quad \text{and} \quad E_{j-1+i}^{j-1}(0,0,i) = C_{j-1+i}(i-1,j-1)$$

holds, i.e.

$$a_{i,j}(z) = \frac{(-1)^{j+i}C_{j-1+i}(i-1,j-1)(2j-1)!(2i-1)!}{16^{j-1+i}(j-1+i)!} z^{-(j-1+i)}\,(1+O(|z|^{-1})) \ . \tag{2.29}$$

Secondly suppose $i=j$. Then, by definition of $\mathbf{A}$, the Laurent series $a_{i,i}(z)$ is obtained for $t=i-1$, i.e. we have

$$a_{i,i}(z) = \sum_{0\le h\le 2i-1} \frac{(-1)^h}{16^h\, h!}\, m_h^{i-1}(i,i)\, z^{-h} + \sum_{2i-1<h} \frac{(-1)^h}{16^h\, h!}\, m_h^{i-1}(i,i)\, z^{-h} \ . \tag{2.30}$$

Lemma 2 (b) implies

$$m_h^{i-1}(i,i) = -((2i-1)!)^2 \sum_{0\le a+b\le h-(2i-1)} D_h^{i-1}(a,b)\, i^{2a} i^{2b}.$$

In particular for $h < 2i-1$ we have $m_h^{i-1}(i,i) = 0$, thus for $\sum_{0\le h\le 2i-1} \frac{(-1)^h}{16^h\, h!}\, m_h^{i-1}(i,i)\, z^{-h}$ only the index $h = 2i-1$ has a nonzero coefficient in which case

$$a = b = 0 \quad \text{and} \quad D_{2i-1}^{i-1}(0,0) = C_{2i-1}(i-1,i-1)$$

holds. We therefore derive

$$a_{i,i}(z) = \frac{(-1)^{2i}((2i-1)!)^2 C_{2i-1}(i-1,i-1)}{16^{2i-1}(2i-1)!} z^{-(2i-1)}(1+O(|z|^{-1})). \tag{2.31}$$

Thus we have proved that we have for $i \le j$

$$a_{i,j}(z) = \frac{(-1)^{j+i} C_{j-1+i}(i-1,j-1)(2j-1)!(2i-1)!}{16^{j-1+i}(j-1+i)!} z^{-(j-1+i)}(1+O(|z|^{-1}))\ . \tag{2.32}$$

Claim 1.

$$C_{j-1+i}(i-1,j-1) = \frac{j\Gamma(2i+2j-1)}{\Gamma(2j+1)\Gamma(2i)} 4^{j+i}. \tag{2.33}$$

According to eq. (2.14)

$$m_h(i,j) \quad = \quad -ij \sum_{0\le a+b\le h-1} C_h(a,b) i^{2a} j^{2b}$$

from which we can conclude for $a+b = h-1$ and $l_m = \min\{a,b\}$

$$\begin{aligned} C_h(a,b) &= \sum_{s=0}^{l_m} \binom{h}{a-s}\binom{h-a+s}{b-s} 4^{a-s} 4^{b-s} 8^{2s+1} 2 \\ &= 4^{h+1} \sum_{s=0}^{l_m} \binom{h}{a-s}\binom{h-a+s}{2s+1} 4^s\ . \end{aligned}$$

Therefore

$$C_{j-1+i}(i-1,j-1) = \frac{j\Gamma(2i+2j-1)}{\Gamma(2j+1)\Gamma(2i)} 4^{j+i} \tag{2.34}$$

and Claim 1 follows. Since $\det[a_{i,j}(z)]_{i,j=1}^{k-1}$ is symmetric, we arrive at

$$a_{i,j}(z) = (-1)^{i+j} \frac{2\Gamma(j+i-\frac{1}{2})}{\sqrt{\pi}} z^{-(j+i-1)}(1+O(|z|^{-1}))$$

for any $1 \le i,j \le k-1$ and the lemma follows. □

Proof of Theorem 1.
Let

$$b_{i,j}(z) = (-1)^{i+j}\frac{2\Gamma(j+i-\frac{1}{2})}{\sqrt{\pi}}z^{-(j+i-1)}\ . \tag{2.35}$$

According to Lemma 3 we have $a_{i,j}(z) = b_{i,j}(z)\,[1+O(|z|^{-1})]$ and we immediately obtain

$$\det[a_{i,j}(z)]_{i,j=1}^{k-1} = \sum_{\sigma\in S_{k-1}}\mathrm{sign}(\sigma)\prod_{j=1}^{k-1}\left[b_{j,\sigma(j)}(z)\ \left[1+O(|z|^{-1})\right]\right]$$

where S_{k-1} denotes the symmetric group over $k-1$ letters. Furthermore we observe

$$\begin{aligned}\det[b_{i,j}(z)]_{i,j=1}^{k-1} &= \sum_{\sigma\in S_{k-1}}\mathrm{sign}(\sigma)\prod_{j=1}^{k-1}b_{j,\sigma(j)}(z)\\ &= \sum_{\sigma\in S_{k-1}}\mathrm{sign}(\sigma)\,(-1)^{\sum_{j=1}^{k-1}(j+\sigma(j))}\left(\frac{2}{\sqrt{\pi}}\right)^{k-1}\times\\ &\qquad z^{-\sum_{j=1}^{k-1}(j+\sigma(j)-1)}\prod_{j=1}^{k-1}\Gamma(j+\sigma(j)-\frac{1}{2})\end{aligned}$$

Since $\sum_{j=1}^{k-1}(j+\sigma(j)) = k(k-1)$ we arrive at

$$\det[b_{i,j}(z)]_{i,j=1}^{k-1} = \left(\frac{2}{\sqrt{\pi}}\right)^{k-1}z^{-(k-1)^2}\ \det\left[\Gamma(j+i-\frac{1}{2})\right]_{i,j=1}^{k-1}$$

and consequently

$$\begin{aligned}\det[a_{i,j}(z)]_{i,j=1}^{k-1} &= \sum_{\sigma\in S_{k-1}}\mathrm{sign}(\sigma)\prod_{j=1}^{k-1}\left[b_{j,\sigma(j)}(z)\ \left[1+O(|z|^{-1})\right]\right]\\ &= \det[\Gamma(j+i-\frac{1}{2})]_{i,j=1}^{k-1}\left(\frac{2}{\sqrt{\pi}}\right)^{k-1}z^{-(k-1)^2}(1+O(|z|^{-1}))\ .\end{aligned}$$

We proceed by computing the determinant

$$\det[\Gamma(j+i-\frac{1}{2})]_{i,j=1}^{k-1} = \prod_{i=1}^{k-1}\Gamma(i+1-\frac{1}{2})\prod_{r=1}^{k-2}r!\ . \tag{2.36}$$

Since $\Gamma(i+j+1-1/2) = (i+j-1/2)\,\Gamma(i+j-1/2)$, we have for $j>1$

$$\Gamma(i+j-\frac{1}{2}) = \prod_{r=1}^{j-1}(i+r-\frac{1}{2})\,\Gamma(i+1-\frac{1}{2})\ .$$

We set

$$u_{i,j}=\begin{cases}\prod_{r=1}^{j-1}(i+r-\frac{1}{2}) & j>1\\ 1 & j=1\end{cases}$$

and compute

$$\det[\Gamma(j+i-\frac{1}{2})]_{i,j=1}^{k-1}=\prod_{i=1}^{k-1}\Gamma(i+1-\frac{1}{2})\det[u_{i,j}]_{i,j=1}^{k-1}=\prod_{i=1}^{k-1}\Gamma(i+1-\frac{1}{2})\det[i^{j-1}]_{i,j=1}^{k-1}.$$

The determinant $\det[i^{j-1}]_{i,j=1}^{k-1}$ is a Vandermonde determinant, whence

$$\det[i^{j-1}]_{i,j=1}^{k-1}=\sum_{1\le i_1<i_2\le k-1}(i_2-i_1)=\prod_{r=1}^{k-2}r!\ .$$

Therefore we have shown

$$\det[a_{i,j}(z)]_{i,j=1}^{k-1}\quad=\quad\left[\prod_{i=1}^{k-1}\Gamma(i+1-\frac{1}{2})\prod_{r=1}^{k-2}r!\right]\left(\frac{2}{\sqrt{\pi}}\right)^{k-1}z^{-(k-1)^2}(1+O(|z|^{-1}))\ . \tag{2.37}$$

It remains to combine our findings: the approximation of the Bessel function eq. (1.7) and eq. (2.6) imply for $-\frac{\pi}{2}<\arg(\mathrm{z})<\frac{\pi}{2}$

$$I_{i-j}(2z)-I_{i+j}(2z)=\frac{e^{2z}}{2\sqrt{\pi z}}\left(\sum_{h=1}^{H}m_h(i,j)\frac{(-1)^h}{16^h h!}\,z^{-h}+O(|z|^{-H-1})\right)\ .$$

Let

$$e_{i,j}^{H}(z)=\sum_{h=1}^{H}m_h(i,j)\frac{(-1)^h}{16^h h!}\,z^{-h} \tag{2.38}$$

then we have

$$F_k(z)=\det[I_{i-j}(2z)-I_{i+j}(2z)]_{i,j=1}^{k-1}=\left(\frac{e^{2z}}{2\sqrt{\pi z}}\right)^{k-1}\left[\det\left[e_{i,j}^{H}\right]_{i,j=1}^{k-1}+O(|z|^{-H-1})\right]. \tag{2.39}$$

Lemma 2 and Lemma 3 provide an interpretation of $\det[e_{i,j}^{H}(z)]_{i,j=1}^{k-1}$: for

$$H>(k-1)^2 \tag{2.40}$$

we can conclude

$$\det[e_{i,j}^{H}(z)]_{i,j=1}^{k-1}=\det[b_{i,j}(z)]_{i,j=1}^{k-1}\left[1+O(|z|^{-1})\right]\ .$$

Accordingly we derive

$$F_k(z)\quad=\quad\left(\frac{e^{2z}}{2\sqrt{\pi z}}\right)^{k-1}\det[b_{i,j}(z)]_{i,j=1}^{k-1}\left[1+O(|z|^{-1})\right]\ .$$

Since

$$\det[b_{i,j}(z)]_{i,j=1}^{k-1} = \left(\frac{2}{\sqrt{\pi}}\right)^{k-1} z^{-(k-1)^2} \det[\Gamma(j+i-\frac{1}{2})]_{i,j=1}^{k-1}$$

and $F_k(z)$ is an even function, we obtain for $\arg(z) \neq \pm\frac{\pi}{2}$

$$F_k(z) = \left[\prod_{i=1}^{k-1}\Gamma(i+1-\frac{1}{2})\prod_{r=1}^{k-2} r!\right]\left(\frac{e^{2z}}{\pi}\right)^{k-1} z^{-(k-1)^2-\frac{k-1}{2}} \; (1+O(|z|^{-1})) \tag{2.41}$$

and the proof of the theorem is complete. □

3. Proof of Theorem 2

Suppose $k = 4m$, $m \in \mathbb{N}$, $p = (k-1)^2 + \frac{k-2}{2} = (4m-1)^2 + 2m - 1$ and

$$g_k(z) = \tilde{c}_k \left[I_0((2k-2)z)\, z^{-p} - \sum_{j=1}^{p} a_{k,j}\, z^{-j}\right], \text{ where } a_{k,j} = [z^{p-j}]\, I_0((2k-2)z). \tag{3.1}$$

For $k = 4m+2$, let $p = (k-1)^2 + \frac{k-2}{2} = (4m+1)^2 + 2m$ and

$$g_k(z) = \tilde{c}_k \left[I_1((2k-2)z)\, z^{-p} - \sum_{j=1}^{p} a_{k,j}\, z^{-j}\right], \text{ where } a_{k,j} = [z^{p-j}]\, I_1((2k-2)z). \tag{3.2}$$

For $k = 4m+1$, let $p = (k-1)^2 + \frac{k-1}{2} = (4m)^2 + 2m$ we set

$$g_k(z) = \tilde{c}_k \left[\cosh((2k-2)z)\, z^{-p} - \sum_{j=1}^{p} a_{k,j}\, z^{-j}\right], \text{ where } a_{k,j} = [z^{p-j}]\, \cosh((2k-2)z). \tag{3.3}$$

Finally, for $k = 4m+3$, let $p = (k-1)^2 + \frac{k-1}{2} = (4m+2)^2 + 2m + 1$ and

$$g_k(z) = \tilde{c}_k \left[\sinh((2k-2)z)\, z^{-p} - \sum_{j=1}^{p} a_{k,j}\, z^{-j}\right], \text{ where } a_{k,j} = [z^{p-j}]\, \sinh((2k-2)z). \tag{3.4}$$

The functions given in eq. (3.1)-(3.4) are entire, even and the constants $\tilde{c}_k$ satisfy

$$g_k(|z|) \sim c'_k\, e^{(2k-2)|z|}\, |z|^{-(k-1)^2-\frac{k-1}{2}}, \qquad \text{as } |z| \to \infty$$

where $c'_k = \pi^{-(k-1)} \prod_{i=1}^{k-1} \Gamma(i+1-\frac{1}{2}) \prod_{r=1}^{k-2} r!$.

Proof. Claim 1. Suppose $z \in \mathbb{C} \setminus \mathbb{R}$, then we have

$$|F_k(z)| = o(|z|^{-1} F_k(|z|)). \tag{3.5}$$

To prove Claim 1, we conclude from Theorem 1 that

$$F_k(z) = c'_k\, e^{(2k-2)z}\, z^{-(k-1)^2 - \frac{k-1}{2}} (1 + O(|z|^{-1})) \quad \text{for } \arg(z) \neq \pm\pi/2 \ , \tag{3.6}$$

where $c'_k = \pi^{-(k-1)} \prod_{i=1}^{k-1} \Gamma(i + 1 - \frac{1}{2}) \prod_{r=1}^{k-2} r!$ holds. We write $z = re^{i\theta}$ and obtain for $\theta \neq 0,\ \pi,\ \pm\pi/2$

$$\frac{|F_k(z)|}{|z|^{-1} F_k(|z|)} \quad = \quad e^{-2(k-1)(1-\cos\theta)\, r}\ r\ \left(O(1) + O(|z|^{-1})\right). \tag{3.7}$$

Therefore we have $|F_k(z)| = o(|z|^{-1} F_k(|z|))$ for $\arg(z) \neq 0, \pi, \pm\pi/2$. Since $|F_k(z)|$ and $|z|^{-1} F_k(|z|)$ are continuous, eq. (3.7) implies

$$|F_k(z)| \quad = \quad o(|z|^{-1} F_k(|z|)), \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R}. \tag{3.8}$$

whence Claim 1.

Claim 2. For any $k \geq 2$, the functions given in eq. (3.1)-(3.4) satisfy

$$|g_k(z)| = o(|z|^{-1} g_k(|z|)) \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R} \tag{3.9}$$

and

$$g_k(|z|) = c'_k\, e^{(2k-2)|z|} |z|^{-(k-1)^2 - \frac{k-1}{2}} (1 + O(|z|^{-1})).$$

Suppose first $k = 4m$ or $4m + 2$. Then we have

$$g_k(z) = \tilde{c}_k \left(I_s((2k-2)z) z^{-p} - \sum_{j=1}^{p} a_{k,j} z^{-j} \right), \quad s = 0 \text{ or } 1, \tag{3.10}$$

where $p = (k-1)^2 + \frac{k-2}{2}$. For $-\frac{\pi}{2} < \arg(z) < \frac{\pi}{2}$, we have

$$I_s(z) = \frac{e^z}{\sqrt{2\pi z}} \left(\sum_{h=0}^{H} \frac{(-1)^h}{h! 8^h} \prod_{t=1}^{h} (4s^2 - (2t-1)^2) z^{-h} + O(|z|^{-H-1}) \right). \tag{3.11}$$

Using eq.(3.11) we derive for sufficiently large $|z|$

$$\begin{aligned} \frac{|g_k(z)|}{|z|^{-1} g_k(|z|)} &\leq \frac{|I_s((2k-2)z)||z|^{-p} + \sum_{j=1}^{p} a_{k,j} |z|^{-j}}{I_s((2k-2)|z|)|z|^{-p-1} - \sum_{j=1}^{p} a_{k,j} |z|^{-j-1}} \\ &\leq C_0\, e^{-2(k-1)(1-\cos\theta) r}\ r \ \ , \end{aligned}$$

where $C_0 > 0$ is some constant. Since $g_k(z)$ is even we have shown

$$|g_k(z)| = o(|z|^{-1} g_k(|z|)) \quad \text{where } \arg(z) \notin \{0, \pi, \frac{\pi}{2}, -\frac{\pi}{2}\}. \tag{3.12}$$

Since $g_k(z)$ is continuous eq. (3.12) implies $|g_k(z)| = o(|z|^{-1} g_k(|z|)) \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R}$. By eq. (3.11) and the definition of $g_k(z)$, we can obtain that

$$\begin{aligned} g_k(|z|) &= \tilde{c}_k \left(I_s((2k-2)|z|)|z|^{-p} - \sum_{j=1}^{p} a_{k,j} |z|^{-j} \right) \\ &= \tilde{c}_k \frac{e^{(2k-2)|z|}}{2\sqrt{(k-1)\pi}\, |z|^{p+\frac{1}{2}}} (1 + O(|z|^{-1})) - \tilde{c}_k \sum_{j=1}^{p} a_{k,j} |z|^{-j} \\ &= c'_k\, e^{(2k-2)|z|} |z|^{-(k-1)^2 - \frac{k-1}{2}} (1 + O(|z|^{-1})). \end{aligned}$$

For $k = 4m+1$ or $4m+3$, $g_k(z)$ satisfies

$$\begin{aligned} |g_k(z)| &\leq \tilde{c}_k \left(\frac{|e^{(2k-2)z}| + |e^{-(2k-2)z}|}{2} |z|^{-p} + \sum_{j=1}^{p} a_{k,j} |z|^{-j} \right) \\ &= \tilde{c}_k \left(\frac{e^{(2k-2)r\cos\theta} + e^{-(2k-2)r\cos\theta}}{2} r^{-p} + \sum_{j=1}^{p} a_{k,j} r^{-j} \right) \end{aligned}$$

where $p = (k-1)^2 + \frac{k-1}{2}$ and consequently for sufficiently large $|z|$

$$\frac{|g_k(z)|}{|z|^{-1} g_k(|z|)} \leq C_1\, r\, e^{-(2k-2)r(1-|\cos\theta|)} \tag{3.13}$$

for some $C_1 > 0$. eq. (3.13) shows

$$\forall z \in \mathbb{C} \setminus \mathbb{R} \qquad |g_k(z)| = o(|z|^{-1} g_k(|z|)) \ . \tag{3.14}$$

For $k = 4m+1$ we derive

$$\begin{aligned} g_k(|z|) &= \tilde{c}_k \left(\cosh((2k-2)|z|)|z|^{-p} - \sum_{j=1}^{p} a_{k,j} |z|^{-j} \right) \\ &= \tilde{c}_k \left(\frac{e^{(2k-2)|z|} + e^{-(2k-2)|z|}}{2} |z|^{-p} - \sum_{j=1}^{p} a_{k,j} |z|^{-j} \right) \\ &= c'_k\, e^{(2k-2)|z|} |z|^{-(k-1)^2 - \frac{k-1}{2}} (1 + O(|z|^{-1})). \end{aligned}$$

The case $k = 4m + 3$ follows analogously. We can conclude from

$$F_k(z) = c_k'\, e^{(2k-2)z} z^{-(k-1)^2 - \frac{k-1}{2}} (1 + O(|z|^{-1})) \quad \text{for } \arg(z) \neq \pm\pi/2,$$

and

$$g_k(|z|) = c_k'\, e^{(2k-2)|z|} |z|^{-(k-1)^2 - \frac{k-1}{2}} (1 + O(|z|^{-1})).$$

that $F_k(|z|) = g_k(|z|)(1 + O(|z|^{-1}))$ holds. To summarize we have shown

$$\begin{aligned} |F_k(z)| &= o(|z|^{-1} F_k(|z|)) \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R} \\ |g_k(z)| &= o(|z|^{-1} g_k(|z|)) \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R} \\ F_k(|z|) &= g_k(|z|)(1 + O(|z|^{-1})). \end{aligned}$$

We can accordingly conclude that

$$|F_k(z) - g_k(z)| = O(|z|^{-1} g_k(|z|)), \tag{3.15}$$

uniformly for all z with $|z| \geq 1$.

Claim 3. For arbitrary $k \geq 2$ we have

$$f_k(n) \sim c_k\, n^{-(k-1)^2 - \frac{k-1}{2}}\ (2k-2)^{2n} \quad \text{where } c_k > 0\ . \tag{3.16}$$

To prove Claim 3 we compute, using eq. (3.15)

$$\begin{aligned} \left| [z^{2n}]\, (F_k(z) - g_k(z)) \right| &\leq \int_{|z| = \frac{n}{k-1}} \frac{|F_k(z) - g_k(z)|}{|z|^{2n+1}} |\mathrm{d}z| \\ &\leq c \int_{|z| = \frac{n}{k-1}} \frac{|z|^{-1} g_k(|z|)}{|z|^{2n+1}} |\mathrm{d}z|, \end{aligned}$$

where c is a positive constant. For $k = 4m$ or $4m + 2$ we have $p = (k-1)^2 + \frac{k-2}{2}$ and substituting for $g_k(|z|)$

$$\begin{aligned} \left| [z^{2n}]\, (F_k(z) - g_k(z)) \right| &\leq c' \int_{|z| = \frac{n}{k-1}} \frac{|z|^{-1} |z|^{-p-\frac{1}{2}} e^{(2k-2)|z|}}{|z|^{2n+1}} |\mathrm{d}z| \\ &= c'\, e^{(2k-2)\cdot\frac{n}{k-1}} \left(\frac{n}{k-1} \right)^{-2n-2-p-\frac{1}{2}} 2\pi \frac{n}{k-1} \\ &= c''\, e^{2n} (k-1)^{2n} n^{-(2n+p+\frac{3}{2})} \end{aligned}$$

where c', c'' are positive constants. By definition of the Bessel function, see eq. (1.5), (3.10) and using Stirling's formula

$$\begin{aligned}[z^{2n}]\, g_k(z) = \tilde{c}_k\, [z^{2n+p}]\, I_s((2k-2)z) &= \tilde{c}_k\, \frac{(k-1)^{2n+p}}{(n+\frac{p-s}{2})!(n+\frac{p+s}{2})!} \\ &\sim \tilde{c}_k'\, e^{2n}(k-1)^{2n} n^{-(2n+p+1)}.\end{aligned} \tag{3.17}$$

Here s only depends on k and $\tilde{c}_k'$ is a positive constant. Therefore we can conclude

$$[z^{2n}]\, F_k(z) \sim [z^{2n}]\, g_k(z)\ , \tag{3.18}$$

whence

$$\begin{aligned} f_k(n) &= (2n)!\, [z^{2n}]\, F_k(z) \sim (2n)!\tilde{c}_k\, \frac{(k-1)^{2n+p}}{(n+\frac{p-s}{2})!(n+\frac{p+s}{2})!} \\ &\sim c_k\, (2k-2)^{2n} n^{-(k-1)^2-\frac{k-1}{2}}\ . \end{aligned}$$

In case of $k = 4m+1$ or $4m+3$ we have $p = (k-1)^2 + \frac{k-1}{2}$ and compute

$$\begin{aligned} \left|[z^{2n}](F_k(z) - g_k(z))\right| &\le c' \int_{|z|=\frac{n}{k-1}} \frac{|z|^{-1}|z|^{-p} e^{(2k-2)|z|}}{|z|^{2n+1}} |\mathrm{d}z| \\ &= c' e^{(2k-2)\,\frac{n}{k-1}} \left(\frac{n}{k-1}\right)^{-2n-2-p} 2\pi \frac{n}{k-1} \\ &= c'' e^{2n}(k-1)^{2n} n^{-(2n+p+1)} \end{aligned}$$

where c', c'' are positive constants. For $k = 4m+1$ we obtain

$$[z^{2n}]\, g_k(z) = \tilde{c}_k\, [z^{2n+p}]\, \cosh((2k-2)z) = \tilde{c}_k\, \frac{(2k-2)^{2n+p}}{(2n+p)!} \sim \tilde{c}_k'\, e^{2n}(k-1)^{2n} n^{-(2n+p+\frac{1}{2})} \tag{3.19}$$

and for $k = 4m+3$

$$[z^{2n}]\, g_k(z) = \tilde{c}_k\, [z^{2n+p}]\, \sinh((2k-2)z) = \tilde{c}_k\, \frac{(2k-2)^{2n+p}}{(2n+p)!} \sim \tilde{c}_k'\, e^{2n}(k-1)^{2n} n^{-(2n+p+\frac{1}{2})}. \tag{3.20}$$

Since $\left|[z^{2n}]\,(F_k(z) - g_k(z))\right| \le c''\, e^{2n}(k-1)^{2n} n^{-(2n+p+1)}$ eq. (3.19) and (3.20) guarantee

$$[z^{2n}]\, F_k(z) \sim [z^{2n}]\, g_k(z)\ . \tag{3.21}$$

Accordingly we obtain

$$f_k(n) = (2n)!\, [z^{2n}]\, F_k(z) \sim (2n)!\, \tilde{c}_k\, \frac{(2k-2)^{2n+p}}{(2n+p)!} \sim c_k\, n^{-(k-1)^2-\frac{k-1}{2}}\, (2k-2)^{2n} \tag{3.22}$$

and Theorem 2 follows. □

Acknowledgments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, the Ministry of Science and Technology, and the National Science Foundation of China.

References

1. M. Abramowitz and I.A. Stegun, eds. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, NBS Applied Mathematics Series 55, National Bureau of Standards, Washington, DC (1964).
2. W.Y.C. Chen, E.Y.P. Deng, R.R.X. Du, R.P. Stanley and C.H. Yan, *Crossings and Nestings of Matchings and Partitions*, Trans. Amer. Math. Soc. **359**, 1555-1575, (2007).
3. W.Y.C. Chen, J. Qin, and C.M. Reidys, *Crossings and Nestings in Tangled Diagrams*, preprint. Available from `arXiv:0710.4053`.
4. G.P. Egorychev, *Integral Representation and the computation of combinatorial sums*, Translations of mathematical monographs American Mathematical Society **59**.
5. I.M. Gessel, *Symmetric functions and P-recursiveness* J.Comb. Theory Ser. A. **53**, 257-285, (1990).
6. I.M. Gessel and D. Zeilberger, *Random Walk in a Weyl chamber*, Proc. Amer. Math. Soc. **115**, 27-31, (1992).
7. D. Grabiner and P. Magyar, *Random Walks in a Weyl Chamber and the decomposition of tensor powers*, J. Alg. Combinatorics **2**, 239-260, (1993).
8. D. Henrion and M. Sebek, *Improved Polynomial Matrix Determinant Computation*, IEEE Trans. CAS-PT I. Fundam. Theory Appl. 46, 1307-1308, (1999).
9. A.M. Odlyzko, *Asymptotic enumeration methods*, Handbook of combinatorics Vol. **2**, 1021-1231, (1995).
10. A. Regev, *Asymptotic values for degrees associated with strips of Young diagrams*, Adv. Math. Vol. **41**, 115-136, (1981).
11. R. Stanley, *Differentiably Finite Power Series*, Europ. J. Combinatorics **1**, 175-188, (1980).
12. P.M. Van Dooren, P. Dewilde and J. Vandewalle, *On the Determination of the Smith-MacMillan Form of a Rational Matrix From Its Laurent Expansion*, IEEE Trans on Circuits and Systems, Vol. **26**, No. 3, 180-189, (1979).
13. D. Zeilberger, *A Holonomic systems approach to special functions identities*, J. of Computational and Applied Math. **32**, 321-368, (1990).

Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071, P.R. China, Phone: *86-22-2350-6800, Fax: *86-22-2350-9272

E-mail address: `reidys@nankai.edu.cn`