1. Lecture 6: Branching Processes

Theorem 1. *Let* $m = \mathbb{E}(\xi) < \infty$ *then there are the following three cases:*
(a) If $m < 1$ *then* $\lim_{n\to\infty} \mathbb{P}(Z_n > 0) = 0$
(b) If $m = 1$ *then* $\lim_{n\to\infty} \mathbb{P}(Z_n > 0) = 0$
(c) If $m > 1$ *then* $\lim_{n\to\infty} \mathbb{P}(Z_n > 0) > 0$.

Proof. Let $f(x) = \sum_{k=0} p_k x^k$, we compute

$$\begin{aligned}
\mathbb{P}(Z_{n+1} = 0) &= \sum_{k=0}^{\infty} \mathbb{P}(Z_1 = k)\mathbb{P}(Z_{n+1} = 0 \mid Z_1 = k) && (1.1)\\
&= \sum_{k=0}^{\infty} \mathbb{P}(Z_1 = k) \underbrace{\mathbb{P}(Z_n = 0)^k}_{\mathbb{P}(Z_{n+1}=0|Z_1=1)^k} && (1.2)\\
&= f(\mathbb{P}(Z_n = 0)) && (1.3)
\end{aligned}$$

For eq. (**??**) we conditioned on the first generation and for eq. (**??**) we used independence of the processes, i.e. conditioned on $Z_1 = k$ the r.v. Z_{n+1} is obtained from k independent branching processes of exactly n (!) generations. Hence we have

$$\mathbb{P}(Z_{n+1} = 0 \mid Z_1 = k) = \mathbb{P}(Z_n = 0)^k \ .$$

It is clear that the sequence $(\mathbb{P}(Z_n = 0))_n$ is bounded by 1 and increasing since the probability of 0-offspring in a subsequent generation can only increase. Therefore there exists a limit

$$q = \lim_{n\to\infty} \mathbb{P}(Z_n = 0)$$

and by continuity of f and $\mathbb{P}(Z_{n+1} = 0) = f(\mathbb{P}(Z_n = 0))$ we obtain $f(q) = q$.
f has as a generating function the properties: (a) $f(0) = 0$ and $f(1) = 1$ and $f'(1) = m$
(b) $f''(x) = \sum_{k\geq 2} k(k-1)p_k x^{k-2} \geq 0$, i.e. f is convex and if $p_0 + p_1 < 1$ we have $f''(x) > 0$ on $(0, 1)$.

Suppose $m > 1$, $p_0 > 0$. Then $g(x) = f(x) - x$ is continuous and has the properties $g(0) > 0$ and $g(1) = 0$. Indeed, $g(x)$ is negative in some neighborhood of 1 since by assumption $g'(1) = m-1 > 0$ and

$$0 < g'(1) = \lim_{\epsilon\to 0} \frac{g(1) - g(1-\epsilon)}{\epsilon} = \frac{-g(1-\epsilon)}{\epsilon}$$

i.e. $g(1-\epsilon) < 0$. As a result there exists some $\alpha \in (0, 1)$ such that $g(\alpha) = 0$ i.e. $f(\alpha) = \alpha$. ☐

Lemma 1. *Suppose* $\lambda_n = \frac{1+\chi_n}{n}$. *Then each* Γ_n*-vertex is contained in a (sc) of size at least* $\lfloor \frac{\chi_n n}{4} \rfloor$ *with probability* $q_0 > 0$*.*

Proof. We consider a GW-process in the sub-cube $Q_2^{n-(k+1)\nu_n}$ (eq. (**??**)). First, by definition $n-(k+1)\nu_n = n-(k+1)\lfloor \frac{\chi_n n}{2(k+1)} \rfloor \geq n - \frac{1}{2}\chi_n n$. W.l.o.g. we initialize this process at $v = (0,\dots,0)$ and set $E_0 = \{e_{n-(k+1)\nu_n+1},\dots,e_n\}$ and $L_0^{(0)} = \{(0,\dots,0)\}$. We consider the $n - \lfloor \frac{3}{4}\chi_n n \rfloor$ smallest neighbors of v. Starting with the smallest we select each of them with independent probability $\frac{1+\chi_n}{n}$. Suppose $v+e_j$ is the first being selected. Then we set $E_1 = E_0 \setminus \{e_j\}$ and $L_1^{(0)} = L_0^{(0)} \cup \{e_j\}$ and proceed inductively setting $E_s = E_{s-1} \setminus \{e_w\}$ and $L_t^{(0)} = L_{t-1}^{(0)} \cup \{e_w\}$ for each neighbor $v+e_w$ being selected until either
(a) all smallest $n - \lfloor \frac{3}{4}\chi_n n \rfloor$ neighbors of $(0,\dots,0)$ are checked or
(b) we have $|E_s| = n - \lfloor \frac{3}{4}\chi_n n \rfloor$, (that is $s = \lfloor \frac{3}{4}\chi_n n \rfloor$). Since

$$n-(k+1)\nu_n - (\lfloor \frac{1}{4}\chi_n n \rfloor - 1) \geq n - \frac{1}{2}\chi_n n - \frac{1}{4}\chi_n n + 1 \geq n - \lfloor \frac{3}{4}\chi_n n \rfloor$$

at least $\lfloor \frac{1}{4}\chi_n n \rfloor - 1$ vertices were connected.
In case of (a) and $|E_s| > n - \lfloor \frac{3}{4}\chi_n n \rfloor$ we choose the smallest element of $L_{t_0}^{(0)}$, v_1^* and set $L_0^{(1)} = L_{t_0}^{(0)} \setminus \{v_1^*\}$. v_1^* has at least $n - \lfloor \frac{3}{4}\chi_n n \rfloor$ neighbors of the form $v_1^* + e_r$ $e_r \in E_s$. We begin with the smallest of these neighbors and continue selecting with probability $\frac{1+\chi_n}{n}$ setting $E_s = E_{s-1} \setminus \{e_j\}$ and $L_t^{(1)} = L_{t-1}^{(1)} \cup \{v_1^* + e_j\}$ for each neighbor $v_1^*+e_j$ being selected. We continue inductively setting $L_0^{(r)} = L_{t_{r-1}}^{r-1} \setminus \{v_r^*\}$ in case of (a) and stop in case of (b). Clearly, this process yields an induced sub-tree of $Q_2^{n-(k+1)\nu_n}$. Suppose we have case (b) and $|E_s| = n - \lfloor \frac{3}{4}\chi_n n \rfloor$ i.e. at least $\lfloor \frac{1}{4}\chi_n\, n \rfloor - 1$ vertices have been connected. Then there are still $n - \lfloor \frac{3}{4}\chi_n n \rfloor$ elements $e_h \in E_{n-\lfloor \frac{3}{4}\chi_n n \rfloor}$. Since we have

$$\frac{1+\chi_n}{n}(n - \lfloor \frac{3}{4}\chi_n n \rfloor) \geq \frac{1+\chi_n}{n}(1 - \frac{3}{4}\chi_n)n = 1 + \frac{1}{4}(1-3\chi_n)\chi_n > 1$$

for $\chi_n < 1/3$, Theorem **??** guarantees that we have a Γ_n-(sc) of size at least $\lfloor \frac{1}{4}\chi_n n \rfloor$ vertices with probability $q_0 > 0$. □